Finite Differences

- If you *absolutely* have to take num. derivatives, what could you do?
  - Compute interpolation coefficients, differentiate basis
  - 'Finite Differences'

2nd row of D

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3) + \alpha_4 f(x_4)$$

$$\approx f'(x_2)$$

$$\underbrace{V' V^{-1}}_{D} \vec{y}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ Right-facing differences

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \cdots$$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\cancel{f(x)} + f'(x)h + f''(x)\frac{h^2}{2} + \cdots - \cancel{f(x)}\right]$$

$$= f'(x) + \underbrace{f''(x)\cdot\frac{h}{2} + \cdots}_{\text{first order accurate.}}$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$ Centered finite differences

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + O(\underbrace{h^2})$$

Demo: Finite Differences vs Noise
Demo: Floating point vs Finite Differences

8.3 Richardson Extrapolation

- If we have two estimates of something, can we get a third that's more accurate? Suppose we have an approximation

$$F = \tilde{F}(h) + O(h^p)$$

and we know $\tilde{F}(h_1)$ and $\tilde{F}(h_2)$.

usually $p+1$

$$F = \tilde{F}(h) + a \cdot h^p + O(h^q)$$

$$F = \alpha \tilde{F}(h_1) + \beta \tilde{F}(h_2) + O(h^q)$$

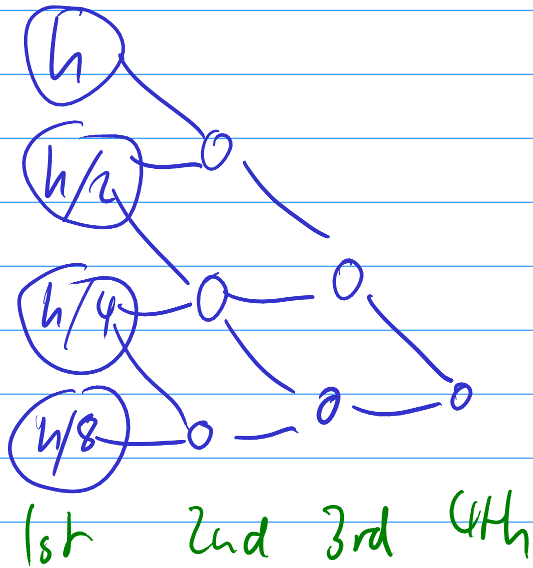
$$\alpha a h_1^p + \beta a \cdot h_2^p = 0 \quad \leftarrow \text{to cancel term}$$

For accuracy: $\alpha + \beta = 1 \quad \rightsquigarrow \quad \beta = 1 - \alpha$

$$\alpha a h_1^p + (1-\alpha) a \cdot h_2^p = 0$$

$$\alpha \cancel{a}(h_1^p - h_2^p) + \cancel{a} \cdot h_2^p = 0$$

$$\alpha = \frac{-h_2^p}{h_1^p - h_2^p}$$



This applies to anything

When applied to quadrature, this is called

Romberg Quadrature.

Demo: Richardson with Finite Differences

9 Initial Value Problems for ODEs

What can we solve?

- Linear Systems: yes ✓
- Nonlinear systems: yes ✓
- Systems with derivatives: no

Systems with derivatives

- ODEs (∂ in 1D)
  - Initial Value Prob. (9)
  - Boundary Value Prob. (10)
- PDEs (∂ in nD) (11)

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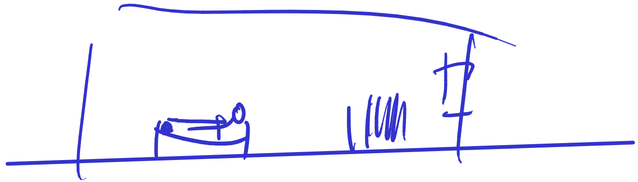
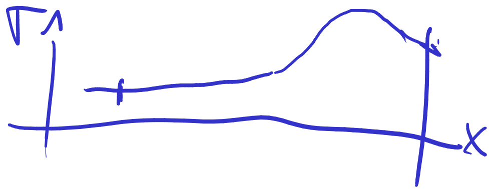
Some Applications

~~ODEs~~ IVP	~~IVPs~~ BVPs
• Population dynamics $y_1' = -\alpha y_2$ (prey) $y_2' = \beta y_1$ (predator) • chemical reactions • equations of motion	• bridge load • pollutatnt concentration (steady state) • temperature (steady state)

$F(x) = m\, x''$





Initial Value Problems: Problem Statement

- Want: Function $\boldsymbol{y}\colon [0, T] \to \mathbb{R}^n$ so that
  - $\boldsymbol{y}^{(k)}(t) = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k-1)})$ (explicit)

    or
  - $\boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k)}) = \boldsymbol{0}$ (implicit)

  are called explicit/implicit *k*th-order ordinary differential equations (ODEs). Give a simple example.

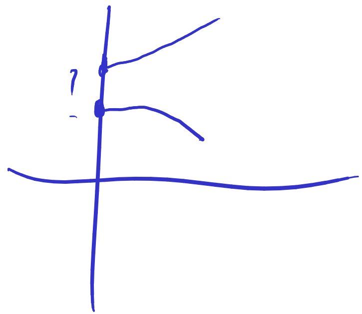
  $y' = \alpha y \quad \to \quad y(x) = e^{\alpha x}$

- Not uniquely solvable on its own. What else is needed?

Need: initial state

Initial condition (IC)

$y(0) = c$



Need: k ICs on $y(t_0), y'(t_0), \ldots y^{(k-1)}(t_0)$

Reducing ODEs to First-Order Form

- A kth order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y)$$