Initial Value Problems: Problem Statement

$y'(t) = \alpha y(t)$

○ Want: Function $\boldsymbol{y}\colon [0, T] \to \mathbb{R}^n$ so that

- $\boldsymbol{y}^{(k)}(t) = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k-1)})$ (explicit)

  or

- $\boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k)}) = \mathbf{0}$ (implicit)

$y(0) = y_0$

$y(t) = y_0 e^{\alpha t}$

are called explicit/implicit kth-order ordinary differential equations (ODEs). Give a simple example.

○ Not uniquely solvable on its own. What else is needed?

Need IC.

$\vec{y}'(t) = \vec{f}(t, \vec{y}(t))$

Reducing ODEs to First-Order Form

- A kth order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'(t) = \begin{pmatrix} y_2 \\ f(y(t)) \end{pmatrix}$$

By introducing enough extra variables, we can convert order to 1.

Properties of ODEs

$$\vec{y}\,'(t) = \vec{f}(t, \vec{y}(t))$$

- What is an autonomous ODE?

$$\vec{y}\,'(t) = \vec{f}(\cancel{t}, \vec{y}(t))$$

- What is a linear ODE?

$$\vec{y}\,'(t) = A(t)\vec{y} + \vec{b}$$

- What is a linear and homogeneous ODE?

$$\vec{y}\,'(t) = A(t)\vec{y}$$

- What is a constant-coefficient ODE?

$$y'(t) = Ay(t)$$

$$y'(t) = 3 + y$$

$$y \to \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}'(t) = \begin{pmatrix} 3\underbrace{u_2}u_1 \\ 1 \end{pmatrix}$$

9.1 Existence, Uniqueness, Conditioning

Existence and Uniqueness

Consider the perturbed problem

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases} \qquad \begin{cases} \hat{\boldsymbol{y}}'(t) = \boldsymbol{f}(\hat{\boldsymbol{y}}) \\ \hat{\boldsymbol{y}}(t_0) = \hat{\boldsymbol{y}}_0 \end{cases}$$

Then if $\boldsymbol{f}$ is Lipschitz continuous (has ‘bounded slope’), i.e.

$$\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\hat{\boldsymbol{y}})\| \leqslant L \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|$$

$$\frac{|f(y) - f(\hat{y})|}{|y - \hat{y}|} \le L$$

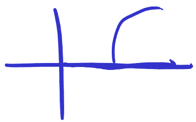
(where L is called the Lipschitz constant), then...

-
- What does this mean for uniqueness?

there exists a solution in a neighborhood of t_0

$$\|y(t) - \hat{y}(t)\| \le C \cdot e^{L(t-t_0)} \|y_0 - \hat{y}_0\|$$

$y'(t) = \sqrt{y}$



Conditioning

Unfortunate terminology accident: "Stability" in ODE-speak
To adapt to conventional terminology, we will use 'Stability' for

- the conditioning of the IVP, *and*
- the stability of the methods we cook up.

Some terminology:

- An ODE is stable if and only if... For all $\varepsilon > 0$

  there exists a $\delta > 0$ $\|y_0 - \hat{y}_0\| < \delta \Rightarrow \|\hat{y}(t) - y(t)\| < \varepsilon$ for all $t \geq t_0$

- An ODE is asymptotically stable if and only if

$$\|\hat{y}(t) - y(t)\| \to 0 \quad (t \to \infty)$$

Example I: Scalar, Constant-Coefficient

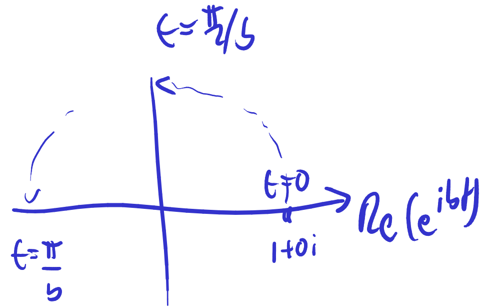
$$\begin{cases} y'(t) = \lambda y \\ y(0) = y_0 \end{cases} \quad \text{where} \quad \lambda = a + ib$$

- Solution?

$y(t) = y_0 e^{\lambda t}$

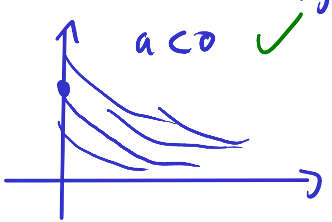
- When is this stable?

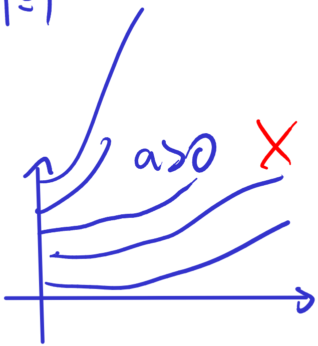
If $\operatorname{Re} \lambda \leq 0$?

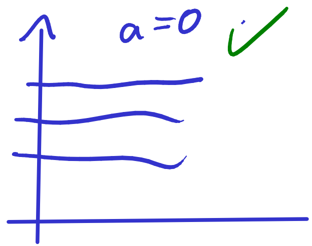


$|y(t)| = |y_0 e^{at} \underbrace{e^{ibt}}_{|\cdot|=1}|$

$= |y_0| |e^{at}|$







$VAV^{-1} = D \qquad B = V^{-1} \rightsquigarrow V = B^{-1} \rightsquigarrow B^{-1}AB = D$

Example II: Constant-Coefficient System

$\vec{y}'(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \vec{y}(t)$

$$\begin{cases} \boldsymbol{y}'(t) = A\boldsymbol{y}(t) \\ \boldsymbol{y}(t) = \boldsymbol{y}_0 \end{cases}$$

Assume $V\cancel{^{-1}}AV^{-1} = D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ diagonal.

$\vec{y} = V^{-1}\vec{w}$

- How do we find a solution?

Define $\vec{w}(t) = V\vec{y}(t)$

$\vec{w}'(t) = V\vec{y}'(t) = VA\vec{y}(t) = \underbrace{VAV^{-1}}_{D}\vec{w}(t)$

$= \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix} \vec{w}(t)$

$w_1'(t) = \lambda_1 w_1(t)$

$\vdots$

- When is this stable?

If $\operatorname{Re} \lambda_i \leq 0$.

(because we can look at stability of $\vec{w}$.)

9.2 Numerical Methods (I)

$$x^2 + 3x - 5 = 0$$

$$y^2 + 3y - 5 = 0$$

Euler's Method

- Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t) = \boldsymbol{y}_0 \end{cases}$$

  - Discrete times: $t_1, t_2, \ldots$, with $t_{i+1} = t_i + h$
  - Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

Demo: Forward Euler stability

9.3 Accuracy and Stability