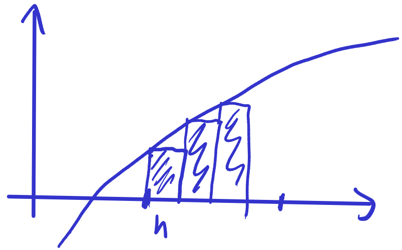
9.2 Numerical Methods (I)

Euler's Method

- Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$



  - Discrete times: $t_1, t_2, \ldots$, with $t_{i+1} = t_i + h$
  - Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

Idea:

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau))\, d\tau$$

$y_0 \quad y_1 \quad y_2 \quad \ldots \quad y_k \quad (y_{k+1})$?

$t_0 \qquad\qquad t_{k+1}$

$$y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(y(\tau))\, d\tau$$

using the left □ rule

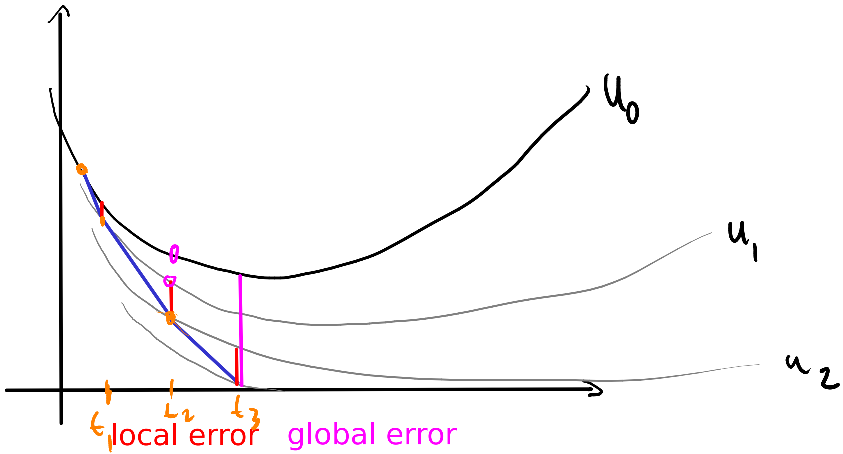
$$y_{k+1} = y_k + h f(y_k)$$

This is called forward Euler.

Demo: Forward Euler stability

9.3 Accuracy and Stability

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$.

- Define the local error at step k as... $l_k = y_k - u_{k-1}(t_k)$
- Define the global error at step k as... $g_k = y_k - y(t_k)$

About Local and Global Error

- Is global error $= \sum$ local errors?

global error $=$ $\sum$ local errors $+$ propagated error

- A time integrator is said to be accurate of order p if...

$$l_k \leq C \cdot h^{p+1}$$

"global error $\geq \sum$ local errors" ← to get to time $t=1$

$$= C \sum_{n=1}^{1/h} h^{p+1}$$

$$= C \cdot \frac{1}{h} \cdot h^{p+1} = C \cdot h^p$$

Stability of a Method

- Find out when forward Euler is stable when applied to

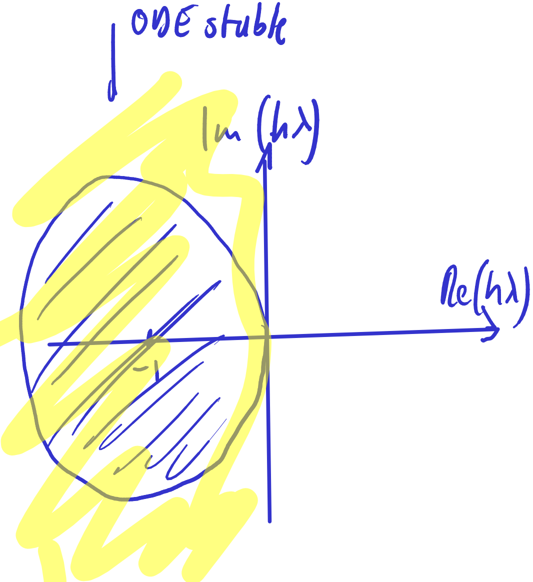
$$y'(t) = \lambda y(t).$$

$$y_{k+1} = y_k + h \cdot \lambda y_k$$

$$= (1 + h\lambda)\, y_k$$

$$= (1 + h\lambda)^{k+1} y_0$$

FW Euler stable $\Leftrightarrow |1 + h\lambda| \leq 1$



Num. example: $\lambda = -1$

$$-2 \leq h\lambda \leq 0$$

$$-2 \leq h(-1)$$

$$h \leq 2$$

Stability: Systems, Nonlinear ODEs

- What about stability for systems, i.e.

$$\boldsymbol{y}'(t) = A\,\boldsymbol{y}(t)?$$

- What about stability for nonlinear systems, i.e.

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t))?$$

Stability for Backward Euler

- Find out when backward Euler is stable when applied to

$$y'(t) = \lambda y(t).$$

Demo: Backward Euler stability