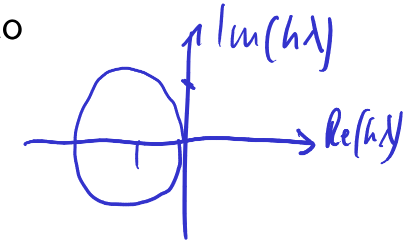
Stability for Backward Euler

FW: $y_{k+1} = y_k + h \cdot f(y_k)$

- Find out when backward Euler is stable when applied to

$$y'(t) = \lambda y(t).$$



BW: $y_{k+1} = y_k + h \cdot f(y_{k+1})$

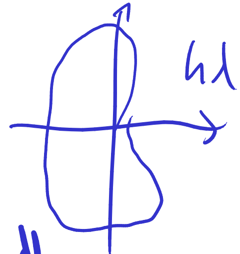
$$y_{k+1} = y_k + h\lambda\, y_{k+1}$$

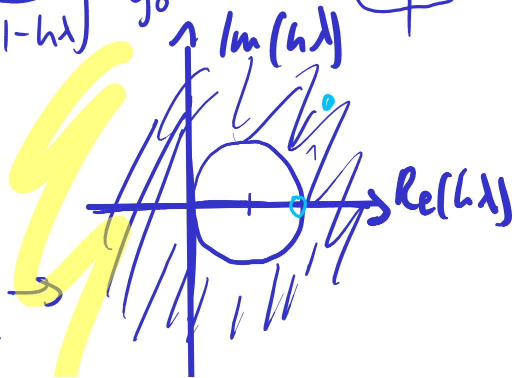
$$y_{k+1}(1 - h\lambda) = y_k$$

$$y_{k+1} = \frac{y_k}{1-h\lambda} = \left(\frac{1}{1-h\lambda}\right)^{k+1} y_0$$

stable $\Leftrightarrow$ $|1 - h\lambda| \geq 1$

$y' = i\lambda$





Stability region for bw Euler

Demo: Backward Euler stability

Stability: Systems, Nonlinear ODEs

$$\begin{pmatrix} -3 & \\ & 1 \end{pmatrix}$$

$w_1'(t) = \lambda_1 w_1(t)$

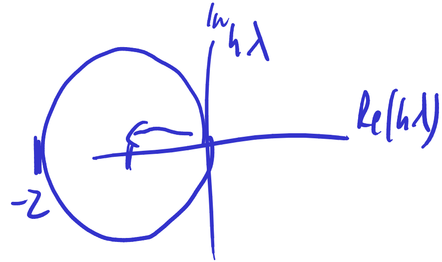
- What about stability for systems, i.e.

$$\mathbf{y}'(t) = A\,\mathbf{y}(t)?$$

$$\rightsquigarrow \quad w'(t) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} w(t)$$

- What about stability for nonlinear systems, i.e.

$$\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t))?$$



Consider perturbation $\begin{cases} \hat{y}'(t) = f(\hat{y}(t)) \\ \hat{y}(0) = \hat{y}_0 \end{cases}$

$h \cdot \lambda$

$h \cdot (-3) = -2$

$h = \frac{2}{3}$

$e(t) = y(t) - \hat{y}(t)$

$$e'(t) = y'(t) - \hat{y}'(t)$$

$$= f(y(t)) - f(\hat{y}(t))$$

$$= J_f\,(y - \hat{y}) = \underline{J_f}\, e$$

$y' = \lambda y$

9.4 Stiffness

$$y' = -100y + 100t + 101$$

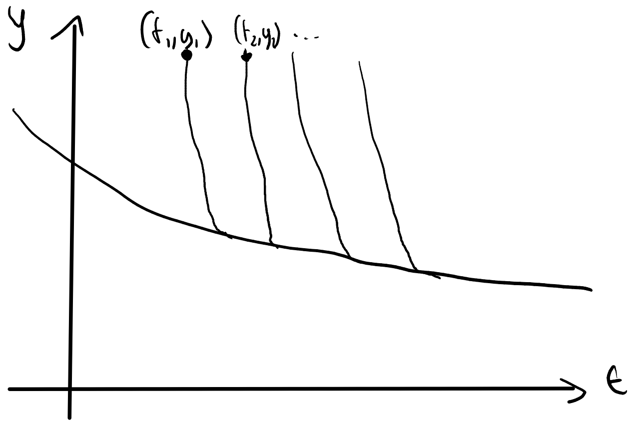
$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$$

$$y_{k+1} = y_k + h(-100y_{k+1} + 100t + 101)$$

$$y_{k+1} = \frac{y_k + 100th + 101h}{(1 + 100h)}$$

Demo: Stiffness

'Stiff' ODEs



- Stiff problems have multiple time scales.
(In the example above: Fast decay, slow evolution.)

- In the case of a stable ODE system

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)),$$

stiffness can arise if $J_{\boldsymbol{f}}$ has eigenvalues of very different magnitude.

- Why not just ‘small’ or ‘large’ magnitude?

- What is the problem with applying explicit methods to stiff problems?

- Phrase this as a conflict between accuracy and stability.

- Can an implicit method take arbitrarily large time steps?

9.5 Numerical Methods (II)

Predictor-Corrector Methods

Idea: Obtain intermediate result, improve it (with same or different method).

For example:

1. 'Predict' with forward Euler: $\tilde{y}_{k+1} = y_k + h\,f(y_k)$
2. 'Correct' with the trapezoidal rule: $y_{k+1} = y_k + \frac{h}{2}(f(y_k) + f(\tilde{y}_{k+1}))$.

This is called Heun's method.

Runge-Kutta/'Single-step'/'Multi-Stage' Methods

Idea: Compute intermediate 'stage values':

$$\begin{aligned} r_1 &= f(t_k + c_1 h, y_k + (a_{11} \cdot r_1 + \cdots + a_{1s} \cdot r_s)h) \\ \vdots & \quad \vdots \\ r_s &= f(t_k + c_s h, y_k + (a_{s1} \cdot r_1 + \cdots + a_{ss} \cdot r_s)h) \end{aligned}$$

Then compute the new state from those:

$$y_{k+1} = y_k + (b_1 \cdot r_1 + \cdots + b_s \cdot r_s)h$$

Can summarize in a Butcher tableau:

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

- When is an RK method explicit?