CS 450: Numerical Anlaysis

Lecture 3
Chapter 2 – Linear Systems
Matrix Norms and Conditioning of Linear Systems

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Vector Norms

- **Properties of vector norms**

$$\|x\| = 0 \iff x = 0$$

$$\|x\| \geq 0$$

$$\|\alpha x\| = |\alpha| \cdot \|x\|$$

$$\|x+y\| \leq \|x\| + \|y\| \quad \text{(triangle inequality)}$$

- **p-norms**

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

Inner-Product Spaces

- **Properties of inner-product spaces**: Inner products $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ must satisfy

properties

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$$
$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \quad \Leftrightarrow \quad \boldsymbol{x} = \boldsymbol{0}$$
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$$
$$\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$$
$$\langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

- **Inner-product-based vector norms**

$\sqrt{\langle x, x \rangle}$

symmetric positive definite matrix A

$\langle x, y \rangle = x^T A y \geq 0$

Matrix Norms

- **Properties of matrix norms:**

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\|A\| \geq 0$$

- **Frobenius norm:** $A_F = \left(\sum_i \sum_j A_{ij}^2\right)^{\frac{1}{2}} = \|vec(A)\|_2$

- **Operator/induced/subordinate matrix norms:**

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$$

Induced Matrix Norms

- **General induced matrix norms:**

$$\|A\|_{m,n} = \max_{\|x\|_n} \|Ax\|_m$$

$$m = n \qquad \|A\|_n$$

- **Interpreting induced matrix norms:**

$$\|A\|_n = \max_{\|x\|_n = 1} \|\underset{y}{Ax}\| \quad \text{maximum amplification}$$

$$\|A^{-1}\|_n = \max_x \frac{\|A^{-1}x\|}{\|x\|} = \max_y \frac{\|y\|}{\|Ay\|} = 1 \Big/ \min_y \frac{\|Ay\|}{\|y\|}$$

Matrix Condition Number

- **Definition**: $\kappa(\boldsymbol{A}) = \underline{||\boldsymbol{A}||} \cdot \underline{||\boldsymbol{A}^{-1}||}$
- **Intuitive derivation**:

$$\kappa(\boldsymbol{A}) = \max_{\text{inputs}} \max_{\text{perturbations in input}} \left| \frac{\text{relative perturbation in output}}{\text{relative perturbation in input}} \right|$$

since a matrix is a linear operator, we can generally decouple its action on the input and the perturbation, so

$$\kappa(\boldsymbol{A}) = \left| \frac{\overbrace{\max_{\text{perturbations in input}} \text{relative perturbation growth}}^{\substack{||A|| \\ \cancel{\kappa(\boldsymbol{A})}}}}{\underbrace{\max_{\text{inputs}} \text{relative input reduction}}_{\substack{\cancel{1/\kappa(\boldsymbol{A}^{-1})} \\ 1/||A^{-1}||}}} \right|$$