CS 450: Numerical Anlaysis

Lecture 4

Chapter 2 – Linear Systems

Orthogonal Matrices and Conditioning of Linear Systems

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Norms and Conditioning of Orthogonal Matrices

- **Orthogonal matrices**: *A matrix $\boldsymbol{Q}$ is orthogonal, if its square and its columns are orthonormal, or equivalently $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$.*
- **Norm and condition number of orthogonal matrices**: *For any $||v||_2 = 1$,*

$$||\boldsymbol{Qv}||_2 = \left(\left\langle \boldsymbol{v}^T\boldsymbol{Q}^T, \boldsymbol{Qv}\right\rangle\right)^{1/2} = \left(\boldsymbol{v}^T\boldsymbol{Q}^T\boldsymbol{Qv}\right)^{1/2} = \left(\boldsymbol{v}^T\boldsymbol{v}\right)^{1/2}$$
$$= ||\boldsymbol{v}||_2$$

Consequently, $||\boldsymbol{Q}||_2 = ||\boldsymbol{Q}^{-1}||_2 = \kappa(\boldsymbol{Q}) = 1$.
$\boldsymbol{Qv}$ expresses $\boldsymbol{v}$ in a coordinate system whose axes are columns of $\boldsymbol{Q}^T$

Singular Value Decomposition

- **The singular value decomposition (SVD):**
  We can express any matrix $\boldsymbol{A}$ *as*

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

  where $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are orthogonal, and* $\boldsymbol{\Sigma}$ *is square nonnegative and diagonal,*

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{max} & & \\ & \ddots & \\ & & \sigma_{min} \end{bmatrix}$$

  Any matrix is diagonal when expressed as an operator mapping vectors from a coordinate system given by $\boldsymbol{V}$ *to a coordinate system given by* $\boldsymbol{U}^T$

Norms and Conditioning using the SVD

- **Norm and condition number in terms of singular values:**
  When multiplying a vector by matrix $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$
  - *Multiplication by* $\boldsymbol{V}^T$ *changes coordinate systems, leaving the norm unchanged*
  - *Multiplication by* $\boldsymbol{U}$ *changes coordinate systems, leaving the norm unchanged*

  so, only multiplication by $\boldsymbol{\Sigma}$ *has an effect on the vector norm*
  - *Note that* $||\boldsymbol{\Sigma}||_2 = \sigma_{max}$, $||\boldsymbol{\Sigma}^{-1}||_2 = \sigma_{min}$, *so*

$$\kappa(\boldsymbol{A}) = \kappa(\boldsymbol{\Sigma}) = \frac{\sigma_{max}}{\sigma_{min}}$$

Conditioning of Linear Systems

- **Lets now return to formally deriving the conditioning of solving $Ax = b$:**
  Consider a perturbation to the right-hand side (input) $\hat{\boldsymbol{b}} = \boldsymbol{b} + \boldsymbol{\delta b}$

$$\boldsymbol{A}\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}$$
$$\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{\delta x}) = \boldsymbol{b} + \boldsymbol{\delta b}$$
$$\boldsymbol{A}\boldsymbol{\delta x} = \boldsymbol{\delta b}$$

  we wish to bound the size of the relative perturbation to the output $||\boldsymbol{\delta x}||/||\boldsymbol{x}||$ *with respect to the size of the relative perturbation the the input* $||\boldsymbol{\delta b}||/||\boldsymbol{b}||$

$$\boldsymbol{\delta x} = \boldsymbol{A}^{-1}\boldsymbol{\delta b}$$
$$\frac{||\boldsymbol{\delta x}||}{||\boldsymbol{x}||} = \frac{||\boldsymbol{A}^{-1}\boldsymbol{\delta b}||}{||\boldsymbol{x}||} \leq \frac{||\boldsymbol{A}^{-1}|| \cdot ||\boldsymbol{\delta b}||}{||\boldsymbol{x}||}$$

  we can use that $||\boldsymbol{x}|| \geq ||\boldsymbol{b}||/\sigma_{max} = ||\boldsymbol{b}||/||\boldsymbol{A}||$ *so*

$$\frac{||\boldsymbol{\delta x}||}{||\boldsymbol{x}||} \leq \underbrace{||\boldsymbol{A}|| \cdot ||\boldsymbol{A}^{-1}||}_{\kappa(\boldsymbol{A})} \cdot \frac{||\boldsymbol{\delta b}||}{||\boldsymbol{b}||} = \frac{\sigma_{max}||\boldsymbol{\delta b}||}{\sigma_{min}||\boldsymbol{b}||}$$

Conditioning of Linear Systems II

- **Consider perturbations to the input coefficients $\hat{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{\delta A}$:**

  In this case we solve the perturbed system

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \boldsymbol{b}$$
$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta A}\boldsymbol{x} = \boldsymbol{b} - \hat{\boldsymbol{A}}\boldsymbol{\delta x}$$
$$\boldsymbol{\delta A}\boldsymbol{x} = -\hat{\boldsymbol{A}}\boldsymbol{\delta x} \approx -\boldsymbol{A}\boldsymbol{\delta x}$$

  we wish to bound the size of the relative perturbation to the output $||\boldsymbol{\delta x}||/||\boldsymbol{x}||$ *with respect to the size of the relative perturbation the the input* $||\boldsymbol{\delta A}||/||\boldsymbol{A}||$

$$\boldsymbol{\delta x} = -\boldsymbol{A}^{-1}\boldsymbol{\delta A}\boldsymbol{x}$$
$$||\boldsymbol{\delta x}|| = ||\boldsymbol{A}^{-1}\boldsymbol{\delta A}\boldsymbol{x}|| \leq ||\boldsymbol{A}^{-1}|| \cdot ||\boldsymbol{\delta A}|| \cdot ||\boldsymbol{x}||$$
$$\frac{||\boldsymbol{\delta x}||}{||\boldsymbol{x}||} \leq \underbrace{||\boldsymbol{A}^{-1}|| \cdot ||\boldsymbol{A}||}_{\kappa(\boldsymbol{A})} \cdot \frac{||\boldsymbol{\delta A}||}{||\boldsymbol{A}||}$$

Solving Simple Linear Systems

- Solve $\boldsymbol{D}\boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{D}$ is diagonal

  $x_i = b_i/d_{ii}$ *with total cost* $O(n)$

- Solve $\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{Q}$ is orthogonal

  $\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{b}$ *with total cost* $O(n^2)$, *given SVD of* $\boldsymbol{A}$ *solve* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, *via solving* $\boldsymbol{U}\boldsymbol{z} = \boldsymbol{b}$ *(orthogonal) then* $\boldsymbol{\Sigma}\boldsymbol{y} = \boldsymbol{z}$ *(diagonal),* $\boldsymbol{V}^T\boldsymbol{x} = \boldsymbol{y}$ *(orthogonal).*

- Solve $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{L}$ is lower-triangular

$$\begin{bmatrix} \boldsymbol{L_{11}} & \boldsymbol{0} \\ \boldsymbol{L_{21}} & \boldsymbol{L_{22}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{bmatrix}$$

  solve recursively $\boldsymbol{L_{11}}\boldsymbol{x_1} = \boldsymbol{b_1}$ *(if* $n = 1$, *divide scalars), then solve recursively for* $\boldsymbol{x_2}$ *in* $\boldsymbol{L_{22}}\boldsymbol{x_2} = \boldsymbol{b_2} - \boldsymbol{L_{21}}\boldsymbol{x_1}$. *The total cost for* $\boldsymbol{L_{11}}$ *of dimension* $n/2$ *is*

$$T(n) = 2T(n/2) + O(n^2) = O(n^2).$$