CS 450: Numerical Analysis
Lecture 19
Chapter 7 Interpolation
Basics of Interpolation

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March 28, 2018
Interpolation

- Given \((t_1, y_1), \ldots, (t_m, y_m)\) with \(t_1 < \cdots < t_m\) an **interpolant** \(f\) satisfies:

  \[
  f(t_i) = y_i \quad \forall i.
  \]

- The interpolant is not unique.

- Error of interpolant can be quantified with knowledge of true function.

- Interpolant is usually constructed as linear combinations of **basis functions** \(\{\phi_j\}_{j=1}^n = \phi_1, \ldots, \phi_n\) so \(f(t) = \sum_j x_j \phi_j(t)\).

  - Interpolant usually doesn’t exist if \(n < m\), exists and is unique if \(n = m\), and is non-unique if \(n > m\).
  - Define \(A = V(t, \{\phi_j\}_{j=1}^n)\) so that \(a_{ij} = \phi_j(t_i)\), then \(Ax = y\).
  - Interpolant can be formed by solving linear system with \(A\) and \(y\).
  - Interpolant can be evaluated at \(t'\) by computing \(B = V(t', \{\phi_j\}_{j=1}^n)\) then \(y' = Bx\).
The choice of monomials as basis functions, \( \phi_j(t) = t^{j-1} \) yields a degree \( n - 1 \) polynomial interpolant:

- Polynomial interpolants yield Vandermonde systems \( A = V(t, \{\phi_j\}_{j=1}^n) \) with \( a_{ij} = t_i^{j-1} \).

- Polynomial interpolants are easy to evaluate and do calculus on:

  \[
  f(t) = x_1 + t(x_2 + t(x_3 + \ldots))
  \]

- Differentiation and integration require \( n \) products.

- Horner’s rule for evaluation requires \( n \) products and \( n \) additions.
Conditioning of Interpolation

- Conditioning of interpolation matrix $A$ depends on basis functions and coordinates $t_1, \ldots, t_m$:
  - $t_i$ defines the $i$th column, so columns tend to be nearly linearly-dependent if $t_i \approx t_{i+1}$
  - $\phi_j$ defines the $j$th row, so rows tend to be nearly linearly-dependent if $\phi_j \in \text{span}(\{\phi_i\}_{i=1,i\neq j}^n)$

- The Vandermonde matrix tends to be ill-conditioned:
  - Monomials of increasing degree increasingly resemble one-another, so rows of $A$ become nearly the same (see demos).
  - The conditioning can be improved somewhat by shifting and scaling points so that each $t_i \in [-1, 1]$. 
Lagrange Basis

- $n$-points fully define the unique $(n - 1)$-degree polynomial interpolant in the Lagrange basis:

$$
\phi_j(t) = \frac{\prod_{k=1, k\neq j}^{n} (t - t_k)}{\prod_{k=1, k\neq j}^{n} (t_j - t_k)}
$$

Note that \textbf{num} is 0 whenever $t = t_k$ for some $k$, while \textbf{den} is never 0, and when $t = t_j$ then \textbf{num} and \textbf{den} are the same, so $\phi_j(t_j) = 1$. Consequently, the resulting interpolant system $V(t, \{\phi_j\}_{j=1}^{n})$ is diagonal.

- Lagrange polynomials yield a convenient Vandermonde system, but the basis functions are hard to evaluate and do calculus on:
  - Evaluation requires $O(n^2)$ work naively and may incur cancellation error.
  - Differentiation and integration are also harder than with monomials.
The Newton basis functions $\phi_j(t) = \prod_{k=1}^{j-1}(t - t_k)$ seek the best of monomial and Lagrange bases:

- Monomials basis functions enable fast evaluation.
- Lagrange basis functions yield a Vandermonde system that's easy to solve.

The Newton basis yields a triangular Vandermonde system:

Note that $a_{ij} = \phi_j(t_i) = 0$ for all $i < j$, so $A$ is lower-triangular. Thus it suffices to use back-substitution to obtain the solution in $O(n^2)$ work.
Recurrences for Newton Basis

- The Newton basis functions \( \phi_j(t) = \prod_{k=1}^{j-1} (t - t_k) \) can be evaluated at \( t \) with \( O(n) \) work using a simple recurrence:

\[
\phi_j(t) = \phi_{j-1}(t)(t - t_j)
\]

- A recurrence known as the divided-differences formula gives a stable way of efficiently computing the coefficients \( x \):

\[
x_i = l_{i1} \quad \text{where} \quad l_{ij} = \frac{l_{i,j+1} - l_{i-1,j}}{t_i - t_j} \quad \text{for} \quad i > j \quad \text{and} \quad l_{ii} = y_i
\]

Matching standard notation we have \( x_i = l_{i1} = f[t_1, \ldots, t_i] \) and generally \( l_{ij} = f[t_j, \ldots, t_i] \). This recurrence also implies that the Newton coefficients can be constructed incrementally by appending new rows to the bottom of \( L \).
Orthogonal Polynomials

Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of *orthonormal functions*:

To compute overlap between basis functions, use a $w$-weighted integral as inner product,

$$\langle p, q \rangle_w = \int_{-\infty}^{\infty} p(t)q(t)w(t)dt$$

Orthonormality is then defined in the usual way, $\{\phi_i\}_{i=1}^{n}$ are orthonormal with respect to the above inner product if

$$\langle \phi_i, \phi_j \rangle_w = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The corresponding norm is given by $||f|| = \sqrt{\langle f, f \rangle_w}$. 
The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:

*Given orthonormal functions* \( \{ \hat{\phi}_i \}_{i=1}^{k-1} \) *obtain* \( k \)th function from \( \phi_k \) via

\[
\hat{\phi}_k(t) = \frac{\psi_k(t)}{||\psi_k||}, \quad \psi_k(t) = \phi_k(t) - \sum_{i=1}^{k-1} \langle \phi_k(t), \hat{\phi}_k(t) \rangle w \hat{\phi}_k(t)
\]

The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with normalization done so \( \hat{\phi}_i(1) = 1 \) and \( w(t) = \begin{cases} 1 : -1 \leq t \leq 1 \\ 0 : \text{otherwise} \end{cases} \)

*For example*, \( \{ \hat{\phi}_i(t) \}_{i=1}^{3} = \{ 1, t, (3t^2 - 1)/2 \} \) since

\[
\psi_1(t) = 1, \quad \psi_2(t) = t - \frac{1}{2} \int_{-1}^{1} t \, dt = t
\]

\[
\psi_3(t) = t^2 - \frac{1}{2} \int_{-1}^{1} t^2 \, dt - t \int_{-1}^{1} t^3 \, dt = t^2 - 1/3
\]