CS 450: Numerical Anlaysis

Lecture 19
Chapter 7 Interpolation
Basics of Interpolation

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Interpolation

- Given $(t_1, y_1), \ldots, (t_m, y_m)$ with $t_1 < \cdots < t_m$ an *interpolant* f satisfies:

$$f(t_i) = y_i \quad \forall i.$$

  - *The interpolant is not unique.*
  - *Error of interpolant can be quantified with knowledge of true function.*

- Interpolant is usually constructed as linear combinations of *basis functions* $\{\phi_j\}_{j=1}^n = \phi_1, \ldots, \phi_n$ so $f(t) = \sum_j x_j \phi_j(t)$.
  - *Interpolant usually doesn't exist if $n < m$, exists and is unique if $n = m$, and is non-unique if $n > m$.*
  - *Define $\boldsymbol{A} = \boldsymbol{V}(\boldsymbol{t}, \{\phi_j\}_{j=1}^n)$ so that $a_{ij} = \phi_j(t_i)$, then $\boldsymbol{Ax} = \boldsymbol{y}$.*
  - *Interpolant can be formed by solving linear system with $\boldsymbol{A}$ and $\boldsymbol{y}$.*
  - *Interpolant can be evaluated at $\boldsymbol{t}'$ by computing $\boldsymbol{B} = \boldsymbol{V}(\boldsymbol{t}', \{\phi_j\}_{j=1}^n)$ then $\boldsymbol{y}' = \boldsymbol{Bx}$.*

Polynomial Interpolation

- The choice of *monomials* as basis functions, $\phi_j(t) = t^{j-1}$ yields a degree $n-1$ polynomial interpolant:
  - *Polynomial interpolants yield Vandermonde systems* $\boldsymbol{A} = \boldsymbol{V}(\boldsymbol{t}, \{\phi_j\}_{j=1}^{n})$ *with* $a_{ij} = t_i^{j-1}$.
- Polynomial interpolants are easy to evaluate and do calculus on:

$$f(t) = x_1 + t(x_2 + t(x_3 + \ldots))$$

  - *Differentiation and integration require* n *products.*
  - *Horner's rule for evaluation requires* n *products and* n *additions*

Conditioning of Interpolation

- Conditioning of interpolation matrix $\boldsymbol{A}$ depends on basis functions and coordinates $t_1, \dots, t_m$:
  - *t_i defines the ith column, so columns tend to be nearly linearly-dependent if $t_i \approx t_{i+1}$*
  - *ϕ_j defines the jth row, so rows tend to be nearly linearly-dependent if $\phi_j \in span(\{\phi_i\}_{i=1,i\neq j}^n)$*
- The Vandermonde matrix tends to be ill-conditioned:
  - *Monomials of increasing degree increasingly resemble one-another, so rows of $\boldsymbol{A}$ become nearly the same (see demos).*
  - *The conditioning can be improved somewhat by shifting and scaling points so that each $t_i \in [-1, 1]$.*

Lagrange Basis

- n-points fully define the unique $(n-1)$-degree polynomial interpolant in the Lagrange basis:

$$\phi_j(t) = \underbrace{\prod_{k=1, k\neq j}^{n} (t - t_k)}_{\textbf{\textit{num}}} \Big/ \underbrace{\prod_{k=1, k\neq j}^{n} (t_j - t_k)}_{\textbf{\textit{den}}}$$

  *Note that **num** is 0 whenever $t = t_k$ for some k, while **den** is never 0, and when $t = t_j$ then **num** and **dem** are the same, so $\phi_j(t_j) = 1$. Consequently, the resulting interpolant system $\boldsymbol{V}(\boldsymbol{t}, \{\phi_j\}_{j=1}^{n})$ is diagonal.*

- Lagrange polynomials yield a convenient Vandermonde system, but the basis functions are hard to evaluate and do calculus on:
  - *Evaluation requires $O(n^2)$ work naively and may incur cancellation error.*
  - *Differentiation and integration are also harder than with monomials.*

Newton Basis

- The Newton basis functions $\phi_j(t) = \prod_{k=1}^{j-1}(t - t_k)$ seek the best of monomial and Lagrange bases:
  - *Monomials basis functions enable fast evaluation.*
  - *Lagrange basis functions yield a Vandermonde system thats easy to solve.*
- The Newton basis yields a triangular Vandermonde system:

  Note that $a_{ij} = \phi_j(t_i) = 0$ for all $i < j$, so $\boldsymbol{A}$ is lower-triangular. Thus it suffices to use back-substitution to obtain the solution in $O(n^2)$ work.

Recurrences for Newton Basis

- The Newton basis functions $\phi_j(t) = \prod_{k=1}^{j-1}(t - t_k)$ can be evaluated at t with $O(n)$ work using a simple recurrence:

$$\phi_j(t) = \phi_{j-1}(t)(t - t_j)$$

- A recurrence known as the divided-differences formula gives a stable way of efficiently computing the coefficients $\boldsymbol{x}$:

$$x_i = l_{i1} \quad \textit{where} \quad l_{ij} = \frac{l_{i,j+1} - l_{i-1,j}}{t_i - t_j} \textit{ for } i > j \quad \textit{and} \quad l_{ii} = y_i$$

matching standard notation we have $x_i = l_{i1} = f[t_1, \ldots, t_i]$ *and generally* $l_{ij} = f[t_j, \ldots, t_i]$*. This recurrence also implies that the Newton coefficients can be constructed* **incrementally** *by appending new rows to the bottom of* $\boldsymbol{L}$*.*

Orthogonal Polynomials

- Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of *orthonormal functions*:

  To compute overlap between basis functions, use a w-weighted integral as inner product,

  $$\langle p, q \rangle_w = \int_{-\infty}^{\infty} p(t)q(t)w(t)dt$$

  orthonormality is then defined in the usual way, $\{\phi_i\}_{i=1}^n$ are orthonormal with respect to the above inner product if

  $$\langle \phi_i, \phi_j \rangle_w = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  the corresponding norm is given by $||f|| = \sqrt{\langle f, f \rangle_w}$.

Legendre Polynomials

- The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:
  Given orthonormal functions $\{\hat{\phi}_i\}_{i=1}^{k-1}$ *obtain* k*th function from* ϕ_k *via*

$$\hat{\phi}_k(t) = \frac{\hat{\psi}_k(t)}{||\hat{\psi}_k||}, \quad \psi_k(t) = \phi_k(t) - \sum_{i=1}^{k-1} \langle \phi_k(t), \hat{\phi}_k(t) \rangle_w \hat{\phi}_k(t)$$

- The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with normalization done so $\hat{\phi}_i(1) = 1$ and $w(t) = \begin{cases} 1 : -1 \leq t \leq 1 \\ 0 : \text{otherwise} \end{cases}$
  For example, $\{\hat{\phi}_i(t)\}_{i=1}^{3} = \{1, t, (3t^2 - 1)/2\}$ *since*

$$\psi_1(t) = 1, \quad \psi_2(t) = t - \frac{1}{2}\int_{-1}^{1} t dt = t$$

$$\psi_3(t) = t^2 - \frac{1}{2}\int_{-1}^{1} t^2 dt - t \int_{-1}^{1} t^3 dt = t^2 - 1/3$$