CS 450: Numerical Analysis

Lecture 20
Chapter 7 Interpolation
Chebyshev Interpolation

Edgar Solomonik

Department of Computer Science
University of Illinois at Urbana-Champaign

April 4, 2018
Orthogonal Polynomials

- Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of orthonormal functions:

\[
\langle f, g \rangle_w = \int_{-\infty}^{\infty} f(x) g(x) \, dx
\]

- \( \|f\|_w = \sqrt{\langle f, f \rangle_w} \) normalization (size measure)

Legendre normalization is by \( f(1) \) end of interval
Legendre Polynomials

The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:

\[
\begin{bmatrix}
1, x, x^2, x^3, \ldots
\end{bmatrix}
\]

\[
e_i(x) = f_i(x) - \sum_{j=1}^{i-1} \langle f_j(x), \hat{e}_j(x) \rangle \hat{e}_j(x)
\]

The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with normalization done so \( \hat{e}_i(1) = 1 \) and \( w(t) = \begin{cases} 1 : & -1 \leq t \leq 1 \\ 0 : & \text{otherwise} \end{cases} \)
Basis Orthogonality and Conditioning

- To obtain perfectly conditioned Vandermonde system, want inner products of different columns to be zero:

\[
A \text{ is orthogonal} \quad \quad A^T A = I
\]

- These inner products should be close to zero, if they are a suitable quadrature rule for our weighted functional inner product:

\[
\int_{-\infty}^{\infty} w(t) \phi_i(t) \phi_j(t) \, dt \approx \sum_{k=1}^{n} \phi_i(t_k) \phi_j(t_k)
\]
Chebyshev Basis

- **Chebyshev polynomials** $\phi_j(t) = \cos(j \arccos(t))$ and **Chebyshev nodes** $t_i = \cos \left( \frac{2i-1}{2n} \pi \right)$ provide a way to pick nodes $t_1, \ldots, t_n$ along with a basis, to yield perfect conditioning:

$$
\phi_j(t^*) = \cos \left( j \frac{2 \mu - 1}{2n} \pi \right) = V_j,
$$

where $w$ is an orthonormal polynomial w.r.t. $w(t) = 1/(1-t^2)$.

$$
\sum_{k=1}^{n} \sum_{j=1}^{n} \phi_j(t_k) \phi_j(t_j) = \sum_{k=1}^{n} \cos \left( j \frac{2 \mu - 1}{2n} \pi \right) \cos \left( j \frac{2 \mu - 1}{2n} \pi \right) = 0, \quad \text{if} \quad i \neq j.
$$

$$
\sum_{k=1}^{n} \phi_j(t_k) = 0,
$$

$$
\forall j.
$$
Chebyshev Nodes Intuition

Note *equi-alteration* property, successive extrema of $T_k$ have opposite sign: and equal magnitude.

Set of $k$ Chebyshev nodes of are given by zeros of $T_k$...

$n$ - nodes get zeros $T$ use basis functions $T_0, \ldots, T_n$ $h = 1, x_i, \ldots$
The Newton polynomials could be obtained by a two-term recurrence.

Legendre and Chebyshev polynomials also satisfy three-term recurrence, for Chebyshev:

\[ \phi_{i+1}(t) = 2t\phi_i(t) - \phi_{i-1}(t) \]

\[ \phi_0(t) = 1 \]
\[ \phi_1(t) = \lambda \]

In fact all orthogonal polynomials satisfy some recurrence of the form:

\[ \phi_{i+1}(t) = (\alpha_i + \beta_i t) \phi_i(t) - \gamma_i \phi_{i-1}(t) \]
Error in Interpolation

Given degree $n$ polynomial interpolant $\tilde{f}$ of $f$, induction on $n$ shows that $E(x) = f(x) - \tilde{f}(x)$ has $n$ zeros $x_1, \ldots, x_n$ and there exist $y_1, \ldots, y_n$ such that

$$E(x) = \int_{x_1}^{x} \int_{y_1}^{w_0} \cdots \int_{y_n}^{w_{n-1}} f^{(n+1)}(w_n)dw_n \cdots dw_0$$

(1)

$$E(x) = E(x_i) + \int_{x}^{x_i} E'(x) \, dx$$

$E(x_i) = 0$

$E(x_i) - E(x_{i-1}) = \int_{x_i}^{x_{i-1}} E'(x) \, dx = 0$

$\exists y_i \in [x_{i-1}, x_i] : E'(y_i) = 0$
Interpolation Error Bounds

- The error bound implies that

\[ |E(x)| \leq \frac{\max_{s \in [x_1, x_n]} |f^{(n+1)}(s)|}{n!} \prod_{i=1}^{n} (x - x_i) \]

for \( x \in [x_1, x_n] \)

- Letting \( h = x_n - x_1 \) (often also achieve same for \( h \) as the node-spacing \( x_{i+1} - x_i \)), we obtain

\[ |E(x)| \leq \frac{\max_{s \in [x_1, x_n]} |f^{(n+1)}(s)|}{n!} h^n = O(h^n) \]

for \( x \in [x_1, x_n] \)

- More nodes \( \rightarrow \) increase \( n \)

- Decreasing \( h \) \( \rightarrow \) higher order convergence

- Motivates use of polynomial approximations
Piecewise Polynomial Interpolation

- The $k$th piece of the interpolant is a polynomial in $[x_i, x_{i+1}]$

- Hermite interpolation ensures consecutive interpolant pieces have same derivative at each knot $x_i$:

\[
\frac{df_i}{dx}(x_{i+1}) = \frac{df_{i+1}}{dx}(x_{i+1})
\]
A **spline** is a \((k - 1)\)-time differentiable piecewise polynomial of degree \(k\):

\[
f_i(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots + \alpha_k x^{k-2} + 2_n x^3
\]

The resulting interpolant coefficients are again determined by an appropriate generalized Vandermonde system:
Lecture 21: Recap Interpolation

Piecewise interpolation

At each knot, both pieces interpolate - smoothness, differentiability.
Vandermonde systems

\[
\begin{bmatrix}
\text{interpolation coefficients}
\end{bmatrix}
= 
\begin{bmatrix}
\text{piece 1 coefficients} \\
\text{piece 2 coefficients} \\
\text{piece 3 coefficients}
\end{bmatrix}
\]

Differentiability

spline interpolant: degree \( k \), \((k-1)\)-time differentiable
natural spline - 2nd derivative at \( x_1, x_n \) is zero
B-Splines

B-splines provide an effective way of constructing splines from a basis:

- The basis functions can be defined recursively with respect to degree.
  \[ e_i^k = v_i^k e_i^{k-1} + (1 - v_i^k) e_{i+1}^{k-1} \]
  \[ v_i^k(t) = \frac{t - t_i}{t_{i+k} - t_i} \]

- The \( i \)th degree \( k \) polynomial piece is positive on \([t_i, t_{i+k+1}]\) and zero everywhere else.

- All possible splines of degree \( k \) with notes \( \{t_i\}_{i=1}^n \) can be represented in the basis.
  So coefficients of neighboring splines are dependent → standard Vandermonde system.