CS 450: Numerical Analysis
Lecture 20
Chapter 7 Interpolation
Chebyshev Interpolation

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Orthogonal Polynomials

Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of orthonormal functions:

To compute overlap between basis functions, use a $w$-weighted integral as inner product,

$$\langle p, q \rangle_w = \int_{-\infty}^{\infty} p(t)q(t)w(t)dt$$

Orthonormality is then defined in the usual way, $\{\phi_i\}_{i=1}^n$ are orthonormal with respect to the above inner product if

$$\langle \phi_i, \phi_j \rangle_w = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The corresponding norm is given by $\|f\| = \sqrt{\langle f, f \rangle_w}$. 
The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis: Given orthonormal functions $\{\hat{\phi}_i\}_{i=1}^{k-1}$ obtain $k$th function from $\phi_k$ via

$$\hat{\phi}_k(t) = \frac{\hat{\psi}_k(t)}{||\hat{\psi}_k||}, \quad \psi_k(t) = \phi_k(t) - \sum_{i=1}^{k-1} \langle \phi_k(t), \hat{\phi}_k(t) \rangle w \hat{\phi}_k(t)$$

The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with normalization done so $\hat{\phi}_i(1) = 1$ and $w(t) = \begin{cases} 1 : & -1 \leq t \leq 1 \\ 0 : & \text{otherwise} \end{cases}$

For example, $\{\hat{\phi}_i(t)\}_{i=1}^3 = \{1, t, (3t^2 - 1)/2\}$ since

$$\psi_1(t) = 1, \quad \psi_2(t) = t - \frac{1}{2} \int_{-1}^{1} tdt = t$$

$$\psi_3(t) = t^2 - \frac{1}{2} \int_{-1}^{1} t^2dt - t \int_{-1}^{1} t^3dt = t^2 - 1/3$$
Basis Orthogonality and Conditioning

- To obtain perfectly conditioned Vandermonde system, want inner products of different columns to be zero:

\[
\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \sum_{k=1}^{n} \phi_i(t_k) \phi_j(t_k) = \delta_{ij}
\]

- These inner products should be close to zero, if they are a suitable quadrature rule for our weighted functional inner product:

\[
\sum_{k=1}^{n} \phi_i(t_k) \phi_j(t_k) \approx \int_{-\infty}^{\infty} w(t) \phi_i(t) \phi_j(t) dt
\]

*For example above holds as equality if we choose*

\[
w(t) = \begin{cases} 
\infty & : t = t_k \text{ for some } k \\
0 & : \text{otherwise}
\end{cases}
\]
Chebyshev Basis

- **Chebyshev polynomials** $\phi_j(t) = \cos(j \arccos(t))$ and **Chebyshev nodes** $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ provide a way to pick nodes $t_1, \ldots, t_n$ along with a basis, to yield perfect conditioning:

  The Chebyshev basis functions are orthonormal with respect to

  $$w(t) = \begin{cases} 
  1/(1 - t^2)^{1/2} & : -1 \leq t \leq 1 \\
  0 & : \text{otherwise}
  \end{cases}.$$

  The Chebyshev nodes ensure orthogonality of the columns of $A$, since

  $$\sum_{k=1}^{n} \phi_l(t_k)\phi_j(t_k) = \sum_{k=1}^{n} \cos\left(\frac{l(2k - 1)}{2n}\pi\right) \cos\left(\frac{j(2k - 1)}{2n}\pi\right)$$

  is zero whenever $j \neq l$ due to periodicity of the summands.
Chebyshev Nodes Intuition

- Note *equi-alteration* property, successive extrema of $T_k$ have opposite sign:
- Set of $k$ Chebyshev nodes are given by zeros of $T_k$
The Newton polynomials could be obtained by a two-term recurrence

Legendre and Chebyshev polynomials also satisfy three-term recurrence, for Chebyshev

\[ \phi_{i+1}(t) = 2t\phi_i(t) - \phi_{i-1}(t) \]

In fact all orthogonal polynomials satisfy some recurrence of the form,

\[ \phi_{i+1}(t) = (\alpha_k t + \beta_k)\phi_i(t) - \gamma_k\phi_{i-1}(t) \]
Error in Interpolation

Given degree $n$ polynomial interpolant $\tilde{f}$ of $f$ induction on $n$ shows that $E(x) = f(x) - \tilde{f}(x)$ has $n$ zeros $x_1, \ldots, x_n$ and there exist $y_1, \ldots, y_n$ such that

$$E(x) = \int_{x_1}^{x} \int_{y_1}^{w_0} \cdots \int_{y_n}^{w_{n-1}} f^{(n+1)}(w_n) dw_n \cdots dw_0$$

(1)

$$E(x) = E(x_1) + \int_{x_1}^{x} E'(w_0) dw_0$$

(2)

Now note that for each of $n - 1$ consecutive pairs $x_i, x_{i+1}$ we have

$$\int_{x_i}^{x_{i+1}} E'(x) dx = E(x_{i+1}) - E(x_i) = 0$$

and so there are $z_i \in (x_i, x_{i+1})$ such that $E'(z_i) = 0$. By inductive hypothesis

$$E'(w_0) = \int_{z_1}^{w_0} \int_{y_2}^{w_1} \cdots \int_{y_n}^{w_{n-1}} f^{(n+1)}(w_n) dw_n \cdots dw_1$$

(3)

Substituting (3) into (2), we obtain (1) with $y_1 = z_1$. 
Interpolation Error Bounds

- The error bound implies that

\[ |E(x)| \leq \frac{\max_{s \in [x_1, x_n]} |f^{(n+1)}(s)|}{n!} \prod_{i=1}^{n} (x - x_i) \quad \text{for} \quad x \in [x_1, x_n] \]

Note that the Choice of Chebyshev nodes decreases this error bound at the extrema, equalizing it with nodes that are in the middle of the interval.

- Letting \( h = x_n - x_1 \) (often also achieve same for \( h \) as the node-spacing \( x_{i+1} - x_i \)), we obtain

\[ |E(x)| \leq \frac{\max_{s \in [x_1, x_n]} |f^{(n+1)}(s)|}{n!} h^n = O(h^n) \quad \text{for} \quad x \in [x_1, x_n] \]

Suggests that higher-accuracy can be achieved by
- adding more nodes (however, high polynomial degree can lead to unwanted oscillations)
- shrinking interpolation interval (suggests piecewise interpolation)
Piecewise Polynomial Interpolation

- The $k$th piece of the interpolant is a polynomial in $[x_i, x_{i+1}]$
  - Typically low-degree polynomial pieces used, e.g. cubic.
  - Degree of piecewise polynomial is the degree of its pieces.
  - Continuity is automatic, differentiability can be enforced by ensuring derivative of pieces is equal at knots (nodes at which pieces meet).

- Hermite interpolation ensures consecutive interpolant pieces have same derivative at each knot $x_i$:

  Hermite interpolation ensures overall interpolant is once differentiable. Various further constraints can be placed on the interpolant if its degree is at least 3, since otherwise the system is underdetermined.
A spline is a \((k - 1)\)-time differentiable piecewise polynomial of degree \(k\):

Cubic splines are different from Hermite cubics

- 2\((n - 1)\) equations needed to interpolate data
- \(n - 2\) to ensure continuity of derivative
- \(n - 2\) to ensure continuity of second derivative for cubic splines

Overall there are 4\((n - 1)\) parameters in the interpolant.

The resulting interpolant coefficients are again determined by an appropriate generalized Vandermonde system:

A natural spline obtains 4\((n - 1)\) constraints by forcing \(f''(x_1) = f''(x_n) = 0\).

Given pieces \(p(x) = \sum_{i=0}^{3} \alpha_i x^i\), \(q(x) = \sum_{i=0}^{3} \beta_i x^i\) the generalized Vandermonde system looks like (3 similar constraints on \(\beta\) not shown)

\[
\begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 & 0 & 0 & 0 & 0 \\
1 & x_2 & x_2^2 & x_2^3 & 0 & 0 & 0 & 0 \\
0 & 1 & 2x_2 & 3x_2^2 & 0 & -1 & -2x_2 & -3x_2^2 \\
0 & 0 & 2 & 6x_2 & 0 & 0 & -2 & -6x_2 \\
0 & 0 & 2 & 6x_1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
0 \\
0 \\
0
\end{bmatrix}
\]