CS 450: Numerical Anlaysis

Lecture 23

Chapter 8 Initial Value Problems for Ordinary Differential Equations

Introduction to Numerical Solutions to ODEs

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Ordinary Differential Equations

$y(t) \in \mathbb{R}^n$

- An *ordinary differential equation (ODE)* usually describes time-varying system by a function $\boldsymbol{y}(t)$ that satisfies a set of equations in its derivatives

$$g(t, y, y', \ldots, y^{(k)}) = 0 \quad \text{implicit}$$

$$y^{(k)} = f(t, y, y', y'', \ldots y^{(k-1)}) \quad \text{explicit}$$

- An ODE of any *order* k can be transformed into a first-order ODE

$y^{(k)}$ appears

1st order $\quad y' = f(t, y)$

$$u = \begin{bmatrix} y \\ \vdots \\ y^{(k-2)} \\ y^{(k-1)} \end{bmatrix}$$

$$u' = f(t, u) = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_k \\ f(t, u) \end{bmatrix}$$

$$u_i = y^{(i-1)}(t)$$

Example: Newton's Second Law

- $F = ma$ corresponds to a second order ODE

$$F(t, y, y') = m\underline{y''} \quad \text{order two}$$

$$y'' = F(t, \underbrace{y, y'}_{u})/m$$

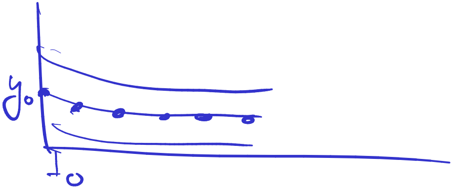
- We can transform it into a first order ODE in two variables

$$u = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad u' = f(t, u) = \begin{bmatrix} u_2 \\ F(t, u)/m \end{bmatrix}$$

Initial Value Problems

- Generally, a first order ODE specifies only the derivative, so the solutions are non-unique, an *initial condition* addresses this:

IVP $\quad y_0 = y(t_0)$



- Given an initial condition, an ODE must satisfy an integral equation for any given point t:

$$y(t) = y_0 + \int_{t_0}^{t} \underbrace{f(s, y(s))}_{y'(s)} \, ds \qquad y' = f(t)$$

Existence and Uniqueness of Solutions

- For an ODE to have a unique solution, it must be defined on a closed domain D and be *Lipschitz continuous*:

Aside
constant coefficients

$$y' = Ay$$
$$= A(t)y \qquad y = e^{At} y$$

Lipschitz continuity

$$\|f(t,y) - f(t,\hat{y})\| \leq L \|y - \hat{y}\|$$

if f is differentiable

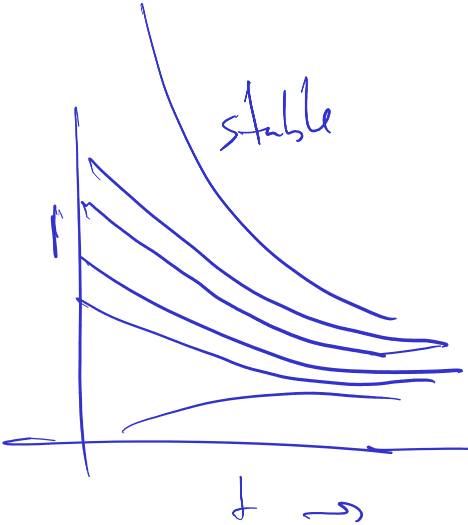
$$L = \max \|J_f\|$$

- The solutions of an ODE can be stable, unstable, or asymptotically stable:

stable if perturbation to y_0 results in perturbation in y that is bounded

otherwise unstable

so asymptotic stability means $\|\hat{y}(t) - y(t)\| \to 0$ as $t \to \infty$

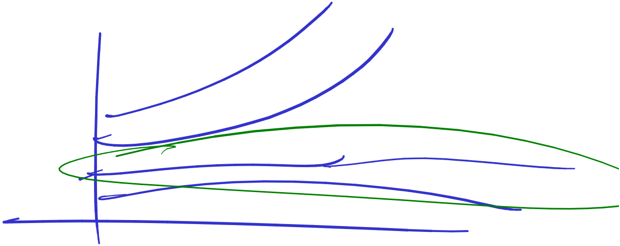


gap → 0

then

asymptotically

stable



Stability of 1D ODEs

- The solution to the scalar ODE $y' = \lambda y$ is $y(t) = y_0 e^{\lambda t}$, with stability dependent on λ:

if $\lambda > 0$ unstable

$\lambda \leq 0$ stable

$\lambda < 0$ asymptotically stable

- A linear ODE generally has the form $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, with stability dependent on the spectral radius (largest eigenvalue) of $\boldsymbol{A}$: constant coefficients

$y(t) = y_0 e^{Dt}$

eigenvalues of A

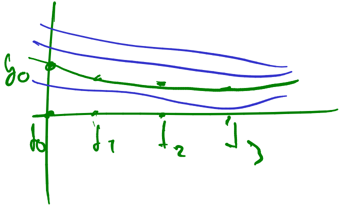
$\lambda_i < 0$ for all i then we have asymptotic stability

Numerical Solutions to ODEs

- Basic methods for numerical ODEs seek to obtain the solution y at a set of points:

discretize the domain

$y(t_k)$ for $\{t_k\}_{k=1}^{m}$



- Euler's method provides the simplest method (attempt) for obtaining a numerical solution:

Forward Euler method

$$y(t_k) = y(t_{k-1}) + h f(t_{k-1}, y(t_{k-1}))$$

Error in Numerical Methods for ODEs

- Truncation error is typically the main quantity of interest, which can be defined *globally* or *locally*

$e(t_k) = y(t_k) - \hat{y}(t_k)$

$e_k = -y(t_k) + \hat{y}_k$

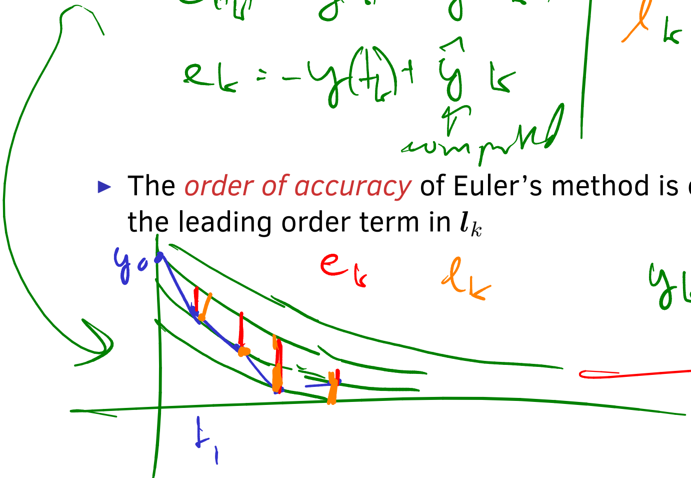
computed

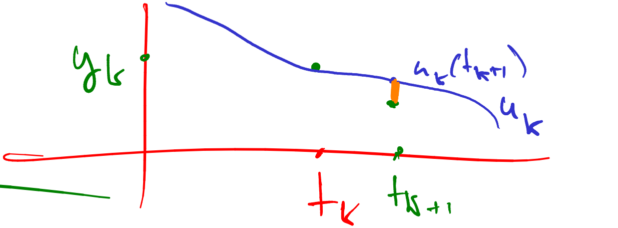
local error

$l_k = \hat{y}_k - u_{k-1}(t_k)$

u_k is true soln to IVP with $y_0(t_k) = \hat{y}_k$

- The *order of accuracy* of Euler's method is one less than than the order of the leading order term in l_k





Stability of Numerical Methods for ODEs

- Stability can be defined for numerical methods similarly to ODEs themselves

defined similarly to stability of problem

even if ODE is stable, method may be unstable

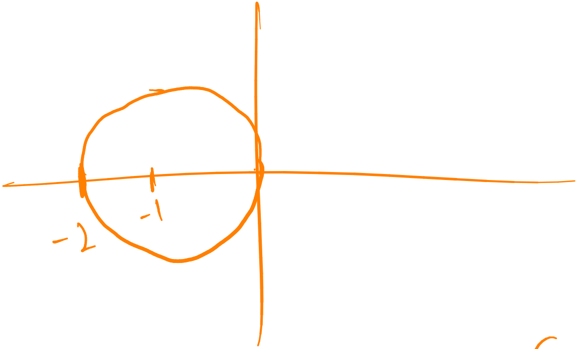
- To analyze stability, can consider application to linear scalar ODE, e.g. for Euler's method:

$$y' = \lambda y \quad \Big| \quad y_k = y_{k-1} + h\underbrace{\lambda y_{k-1}}_{y'(t_{k-1})} = \underbrace{(1+h\lambda)}_{\text{growth factor}} y_{k-1}$$

$$|1+h\lambda| \leq 1$$

$$h\lambda \in [-2, 0] \Rightarrow \text{stability}$$

complex plane



Stability region for Forward Euler

Implicit Methods

- Implicit methods for ODEs form a sequence of solutions that satisfy conditions on a local approximation to the solution:

Backward Euler

$$y_{k+1} = y_k + \underline{hf(t_{k+1}, y_{k+1})}$$

- The backward Euler method for a simple linear scalar ODE stability region is the left half of the complex plane:

$$y' = \lambda y$$

$$y_{k+1} = y_k + h\lambda y_{k+1} \qquad (1-h\lambda) y_{k+1} = y_k$$

$$y_{k+1} = \frac{1}{1-h\lambda} y_k$$

when is $\left|\frac{1}{1-h\lambda}\right| \le 1$? whenever $\lambda \le 0$