CS 450: Numerical Analysis
Lecture 23
Chapter 8 Initial Value Problems for Ordinary Differential Equations
Introduction to Numerical Solutions to ODEs

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An ordinary differential equation (ODE) usually describes time-varying system by a function $y(t)$ that satisfies a set of equations in its derivatives

$$g(t, y, y', \ldots, y^{(k)}) = 0$$

Implicit

$$y^{(k)} = f(t, y, y', y'', \ldots, y^{(k-1)})$$

Explicit

An ODE of any order $k$ can be transformed into a first-order ODE

$y^{(k)}$ appears

$y' = f(t, y)$

$u_1 = y^{(1)}(t)$

$u_2 = y^{(2)}(t)$

$u_n = f(t, u)$
Example: Newton’s Second Law

- $F = ma$ corresponds to a second order ODE

$$F(t, y, y') = my''$$  \text{order two}

$$y'' = \frac{F(t, y, y')}{m}$$

- We can transform it into a first order ODE in two variables

$$u = \begin{bmatrix} y \\ y' \end{bmatrix} \quad u' = f(t, u) = \begin{bmatrix} u_2 \\ F(t, u)/m \end{bmatrix}$$
Initial Value Problems

- Generally, a first order ODE specifies only the derivative, so the solutions are non-unique, an *initial condition* addresses this:

\[ y_0 = y(0) \]

- Given an initial condition, an ODE must satisfy an integral equation for any given point \( t \):

\[ y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds \]

\[ y'(t) = f(t) \]
Existence and Uniqueness of Solutions

- For an ODE to have a unique solution, it must be defined on a closed domain $D$ and be Lipschitz continuous:

  \[ \| f(t, y) - f(t, \tilde{y}) \| \leq L \| y - \tilde{y} \| \]

  if $f$ is differentiable

  $L = \max \| J f \|$

- The solutions of an ODE can be stable, unstable, or asymptotically stable:

  - Stable if perturbation to $y_0$ results in perturbation in $y$ that is bounded
  - Otherwise unstable
  - So asymptotic stability means $\| y(t) - y(0) \| \to 0$ as $t \to 0$
If \( \text{gap} > 0 \) then asymptotically stable.
Stability of 1D ODEs

- The solution to the scalar ODE $y' = \lambda y$ is $y(t) = y_0 e^{\lambda t}$, with stability dependent on $\lambda$:

  - if $\lambda > 0$, unstable
  - $\lambda \leq 0$, stable
  - $\lambda < 0$, asymptotically stable

- A linear ODE generally has the form $y' = Ay$, with stability dependent on the spectral radius (largest eigenvalue) of $A$:

  - constant coefficient
  - $y(t) = y_0 e^{\lambda t}$
  - eigenvalues of $A$

  - if $\lambda_i < 0$ for all $i$, then we have asymptotic stability
Numerical Solutions to ODEs

- Basic methods for numerical ODEs seek to obtain the solution $y$ at a set of points:

  \[ y(t_k) \text{ for } k = 0, 1, \ldots, n \]

  \[ \text{discretize the domain} \]

- Euler’s method provides the simplest method (attempt) for obtaining a numerical solution:

  \[ \text{Forward Euler method} \]

  \[ y(t_{k+1}) = y(t_k) + hf(t_k, y(t_k)) \]
Truncation error is typically the main quantity of interest, which can be defined \textit{globally} or \textit{locally}.

\[ e_{tk} = y(tk) - \hat{y}(tk) \]
\[ e_k = -y(t_k) + \hat{y}(tk) \]

The \textit{order of accuracy} of Euler’s method is one less than the order of the leading order term in \( l_k \).
Stability of Numerical Methods for ODEs

- Stability can be defined for numerical methods similarly to ODEs themselves.

\[ \text{defined similarly to stability of problem} \]

\[ \underline{\text{even if ODE is stable, method may be unstable}} \]

- To analyze stability, can consider application to linear scalar ODE, e.g. for Euler’s method:

\[ y' = -y \quad | \quad y_k = y_{k-1} + h \cdot y_{k-1} = \frac{(1 + h\lambda) y_{k-1}}{1 + h|\lambda|} \]

\[ |1 + h\lambda| \leq 1 \quad \text{for } \lambda \in [-2, 0] \Rightarrow \text{stability} \]
complex plane

stability region for forward Euler
Implicit Methods

Implicit methods for ODEs form a sequence of solutions that satisfy conditions on a local approximation to the solution:

\[
\text{Backward Euler} \quad y_{k+1} = y_k + hf(x_{k+1}, y_{k+1})
\]

The backward Euler method for a simple linear scalar ODE stability region is the left half of the complex plane:

\[
y' = -y \\
y_{k+1} = y_k + h \left(1 - h \right) y_{k+1} = y_k \\
y_{k+1} = y_k + h y_k \\
\text{when } |1 - h| \leq 1 \text{ whenever } x \leq 0 \quad y_{k+1} = \frac{1}{1 - h} y_k
\]