CS 450: Numerical Anlaysis

Lecture 23

Chapter 9 Initial Value Problems for Ordinary Differential Equations

Introduction to Numerical Solutions to ODEs

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Ordinary Differential Equations

- An *ordinary differential equation (ODE)* usually describes time-varying system by a function $\boldsymbol{y}(t)$ that satisfies a set of equations in its derivatives *The general implicit form is*

$$\boldsymbol{g}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \dots, \boldsymbol{y}^{(k)}) = \boldsymbol{0},$$

but we restrict focus on the explicit form, $\boldsymbol{y}^{(k)} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \dots, \boldsymbol{y}^{(k-1)})$.

- An ODE of any *order* k can be transformed into a first-order ODE,

$$\boldsymbol{u}' = \begin{bmatrix} \boldsymbol{u}_1' \\ \vdots \\ \boldsymbol{u}_{k-1}' \\ \boldsymbol{u}_k' \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_k \\ \boldsymbol{f}(t, \boldsymbol{u}_1, \dots, \boldsymbol{u}_k) \end{bmatrix} = \quad \textit{where} \quad \boldsymbol{u}_i(t) = \boldsymbol{y}^{(i-1)}(t).$$

Consequently we restrict our focus to systems of first-order ODEs. Of particular importance are linear ODEs, which have the form $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}$, *whose coefficients are said to be constant if* $\boldsymbol{A}(t) = \boldsymbol{A}$ *for all* t.

Example: Newton's Second Law

- $F = ma$ corresponds to a second order ODE

$$F(t, y(t), y'(t)) = my''(t)$$
$$y''(t) = f(t, y, y') = F(t, y(t), y'(t))/m$$

- We can transform it into a first order ODE in two variables

$$\boldsymbol{u} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u}) = \begin{bmatrix} u_2 \\ F(t, \boldsymbol{u})/m \end{bmatrix}$$

Initial Value Problems

- Generally, a first order ODE specifies only the derivative, so the solutions are non-unique, an *initial condition* addresses this:

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

This condition yields an initial value problem (IVP), which is the simplest example of a boundary condition.

- Given an initial condition, an ODE must satisfy an integral equation for any given point t:

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{y}(s))ds$$

In the special case that $\boldsymbol{y}' = \boldsymbol{f}(t)$, the integral can be computed directly by numerical quadrature to solve the ODE.

Existence and Uniqueness of Solutions

- For an ODE to have a unique solution, it must be defined on a closed domain D and be *Lipschitz continuous*:

$$\forall \boldsymbol{y}, \hat{\boldsymbol{y}} \in D, \quad ||\boldsymbol{f}(t, \hat{\boldsymbol{y}}) - \boldsymbol{f}(t, \boldsymbol{y})|| \leq L||\hat{\boldsymbol{y}} - \boldsymbol{y}||,$$

i.e. the rate of change of the ODE should itself change continuously. Any differentiable function f is Lipschitz continuous with

$$L = \max_{(t,\boldsymbol{y}) \in D} ||\boldsymbol{J}_{\boldsymbol{f}}(t, \boldsymbol{y})||.$$

- The solutions of an ODE can be stable, unstable, or asymptotically stable:

Perturbation to the input causes a perturbation to the solution that has bounded growth for a stable ODE, unbounded for an unstable ODE, and shrinks for an asymptotically stable ODE.

Stability of 1D ODEs

- The solution to the scalar ODE $y' = \lambda y$ is $y(t) = y_0 e^{\lambda t}$, with stability dependent on λ:

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & : \lambda > 0 \text{ (unstable)} \\ y_0 & : \lambda = 0 \text{ (stable)} \\ 0 & : \lambda < 0 \text{ (asymptotically stable)} \end{cases}$$

- A linear ODE generally has the form $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, with stability dependent on the spectral radius (largest eigenvalue) of $\boldsymbol{A}$:

  For general ODEs, stability can be ascertained locally by a considering a linear approximatiaion $\boldsymbol{f}(t, \boldsymbol{y}) \approx \boldsymbol{y} + \boldsymbol{J_f}(t)^{-1}\boldsymbol{y}$ and measuring the spectral radius of $\boldsymbol{J_f}(t)^{-1}$.

Numerical Solutions to ODEs

- Methods for numerical ODEs seek to approximate $\boldsymbol{y}(t)$ at $\{t_k\}_{k=1}^m$.

  Compute $\hat{\boldsymbol{y}}_k$ *for* $k \in \{1, \ldots, m\}$ *so as to approxumate* $\boldsymbol{y}(t_k) \approx \hat{\boldsymbol{y}}_k$. *For an IVP, typically form* $\hat{\boldsymbol{y}}_{k+1}$ *using* $\hat{\boldsymbol{y}}_k$ *or additionally (for multistep methods)* $\hat{\boldsymbol{y}}_{k-1}, \ldots$

- Euler's method provides the simplest method (attempt) for obtaining a numerical solution:

  Approximation solution to ODE at $t_k + h$ *by linear segment from* $(t_k, \boldsymbol{y}_k)$ *with slope* $\boldsymbol{f}(t_k, \boldsymbol{y}_k)$*,*

  $$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + h_k \boldsymbol{f}(t_k, \boldsymbol{y}_k).$$

  Its instructive to observe that this approximation arises as the first order form of various models (Taylor series, finite differences, interpolation, quadrature, undetermined coefficients).

Error in Numerical Methods for ODEs

- Truncation error is typically the main quantity of interest, which can be defined *globally* or *locally*
  Global error is measured at all points

$$\boldsymbol{e}_k = \hat{\boldsymbol{y}}_k - \boldsymbol{y}(t_k),$$

  which local error measures the deviation from the exact solution $\boldsymbol{u}_{k-1}(t)$ *passing through the previous point* $(t_{k-1}, \hat{\boldsymbol{y}}_{k-1})$,

$$\boldsymbol{l}_k = \hat{\boldsymbol{y}}_k - \boldsymbol{u}_{k-1}(t_k).$$

- The *order of accuracy* of Euler's method is one less than than the order of the leading order term in $\boldsymbol{l}_k$
  Accuracy is of order p *if* $\boldsymbol{l}_k = O(h_k^{p+1})$. *Euler's method is first order accurate by Taylor expansion analysis.*

Stability of Numerical Methods for ODEs

- Stability can be defined for numerical methods similarly to ODEs themselves.

  A method for an ODE with stable solutions can be unstable. Usually, we will seek to establish a stability region for a method, which describes the step-size conditions necessary for stability in terms of the step size h and (the largest) eigenvalue λ, usually as a function of $h\lambda$.

- Basic stability properties follow fron analysis of linear scalar ODE, which serves as a local approximation to more complex ODEs.

  Consider forward Euler for the ODE $y' = \lambda y$, where

$$y_{k+1} = y_k + h\lambda y_k = (1 + h\lambda) y_k.$$

  Euler's method requires $|1 + h\lambda| \leq 1$ to be stable, which implies $-2 \leq h\lambda \leq 0$ for real λ. For a general ODE, the eigenvalues of $\boldsymbol{J}_f$ must lie within a circle on the complex plain centered at -1 with radius 1.

Implicit Methods

- Implicit methods for ODEs form a sequence of solutions that satisfy conditions on a local approximation to the solution:

  The most basic implicit method is the backward Euler method

  $$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \boldsymbol{h}_k \boldsymbol{f}(t_{k+1}, \boldsymbol{y}_{k+1}),$$

  which solves for $\boldsymbol{y}_{k+1}$ so that a linear approximation of the solution at t_{k+1} passed through the point $(t_k, \boldsymbol{y}_k)$. Just like forward Euler, first-order accuracy is achieved by the linear approximation.

- The backward Euler method for a simple linear scalar ODE stability region is the left half of the complex plane:

  Such a method is called unconditionally stable. Note that the growth factor can be derived via

  $$y_{k+1} = y_k + h\lambda y_{k+1} = \frac{1}{1 - h\lambda} y_k,$$

  and satisfies $|1/(1 - h\lambda)| \leq 1$ so long as $h\lambda \leq 0$.