Finite Difference Methods

Let's derive the finite difference method for the ODE BVP defined by

\[ u'' + 1000(1 + t^2)u = 0 \]

with boundary conditions \( u(-1) = 3 \) and \( u(1) = -3 \).

\[ u_i \approx u(t_i), \quad \text{for } t_0, \ldots, t_n \quad (t_0 = -1, \quad t_n = 1) \]

Centered F-D approximation for \( u'' \)

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + 1000(1 + t_i^2)u_i = 0 \]

\[
\begin{bmatrix}
1 & -2 & 1 & \cdots & 0 \\
-2 & 5 & -2 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -2 & 1 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n
\end{bmatrix}
+ \begin{bmatrix}
1000(1+t_0^2) \\
1000(1+t_1^2) \\
1000(1+t_2^2) \\
\vdots \\
1000(1+t_n^2)
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u_3 \\
u_4 \\
\vdots \\
u_n-1
\end{bmatrix}
= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \]
Collocation Methods

- **Collocation methods** approximate \( y \) by representing it in a basis

\[
y(t) = v(t, \mathbf{x}) = \sum_{i=1}^{n} x_i \phi_i(t).
\]

- **Spectral methods** use polynomials or trigonometric functions for \( \phi_i \), which are nonzero over most of \([a, b]\), while **finite element** methods leverage basis functions with local support (e.g. B-splines).
Solving BVPs by Optimization

- We reformulate the collocation approximation as an optimization problem:

\[ r(t_i, x) = v(t_i, x) - f(t_i, v(t_i, x)) \]

\[ = \sum_{i=1}^{n} c_i \psi_i(x) - f(t_i, v(t_i, x)) \]

\[ F(x) = \frac{1}{2} \int_{a}^{b} ||r(t, x)||^2 \, dt \]

- The first-order optimality conditions of the optimization problem are a system of linear equations \( Ax = b \):

\[ \frac{df(x)}{dx} = \int_{a}^{b} r(t, x)^T \frac{dr}{dx} (t, x) \, dt \]

\[ = \int_{a}^{b} r(t, x)^T \psi'_i(t) \, dt = \sum_{i=1}^{n} c_i \int_{a}^{b} \psi'_i(t) \psi'_j(t) \, dt - \int_{a}^{b} f(t) \, dt \]

\[ Ax = b \]
Weighted Residual

- **Weighted residual methods** work by ensuring the residual is orthogonal with respect to a given set of weight functions:

\[
0 = \int_a^b r(t, x)^T \psi_i(t) \, dt
\]

\[
0 = \sum_j \int_a^b w_j \psi_j(t)^T \psi_i(t) \, dt - \int_a^b f(t) \psi_i(t) \, dt
\]

- The Galerkin method is a weighted residual method where \( w_i = \phi_i \).
Let's apply the Galerkin method to the more general linear ODE
\[ f(t, y) = A(t)y(t) + b(t) \]
with residual equation,
\[ r = y' - f = y' - A y - b \]
\[ r(t, y) = \sum_j x_j \phi_j(y) - A \sum_j x_j \phi_j(t) - b(t) \]
\[ = \sum_j x_j \left( \phi_j' - A \phi_j(t) \right) - b(t) \]
\[ = \sum_j x_j \int (\phi_j' - A \phi_j(t)) \phi_i(t) \, dt - \int b(t) \phi_i(t) \, dt \]
Nonlinear BVPs: Poisson Equation

In practice, BVPs are at least second order and it's advantageous to work in the natural set of variables.

Consider the Poisson equation $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:

$$\phi_i(t) = \begin{cases} 
(t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\
(t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\
0 & : \text{otherwise}
\end{cases}$$

where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$. 

$r = u'' - f \Rightarrow 
\sum_j \phi''_j (x_j) - f(t) = \{ t_i, t_{i+1} \}$
Weak Form and the Finite Element Method

The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in **weak form**:

If \( \phi_i \) satisfies boundary conditions

\[
\int_a^b f(x) \phi_i(x) \, dx = \int_a^b u''(x) \phi_i(x) \, dx + \int_a^b u'(b) \phi_i(b) - u'(a) \phi_i(a) - \int_a^b u'(x) \phi_i'(x) \, dx
\]

\( u'(a) = \sum \phi_i'(a) \) | \( u'(b) = \sum \phi_i'(b) \)
\[
\int_{a}^{b} (e_i(t) + e_j(t)) \, dt = \begin{cases} 
-3 & \text{if } t \in \mathbb{T}, \\
\frac{1}{h} & \text{otherwise}
\end{cases}
\]

\[
e_i(t) = \begin{cases} 
\frac{t - i \cdot h}{h} & t \in [i - \frac{1}{2}h, i + \frac{1}{2}h] \\
0 & \text{otherwise}
\end{cases}
\]

\[
e_j(t) = \begin{cases} 
\frac{t - j \cdot h}{h} & t \in [j - \frac{1}{2}h, j + \frac{1}{2}h] \\
0 & \text{otherwise}
\end{cases}
\]

\[
A_{ij} = \int_{a}^{b} (e'_i(t) e'_j(t)) \, dt = \frac{1}{2h} \sum_{i=1}^{b} \text{if } |i-j| = 1, \quad 0 \quad \text{otherwise}
\]
Finite Element Methods in Practice

- Hat functions are linear instances of $B$-splines:

  $\text{degree } k, k-1$ times differentiable

- Finite-element methods readily generalize to PDEs:

  FEM with triangles, tetrahedra
A typical second-order scalar BVP eigenvalue problem has the form

\[ u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0 \]

\[ f(t, u, u') = g(t) \cdot u \]

\[ u_i \quad \text{for } i = 1, \ldots, n \]

\[ u_i \approx u_i(t_i) \]

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \lambda g_i \cdot u_i \]

\[ u = \lambda u \]
Eigenvalue Problems with ODEs

- Generalized eigenvalue problems arise from more sophisticated ODEs,

\[ u'' = \lambda (g(t)u + h(t)u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0 \]

\[
\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \lambda \left( g_i u_i + h_i \frac{u_{i+1} - u_{i-1}}{2h} \right)
\]

\[ A_u = \lambda B_u \]