Finite Difference Methods

Let's derive the finite difference method for the ODE BVP defined by

$$u'' + 1000(1 + t^2)u = 0$$

with boundary conditions $u(-1) = 3$ and $u(1) = -3$.

Using a discretization with points $t_1, \ldots, t_n$, $t_{i+1} - t_i = h$, and a centered difference approximation for $u''$ we obtain

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + 1000(1 + t_i)u_i = 0.$$

We can rewrite the above using linear equations with matrices

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1000(1 + t_1) & 0 \\ & \ddots \end{bmatrix}$$

and solve the system $(A + B)u = \begin{bmatrix} 3 & 0 & \cdots & 0 & -3 \end{bmatrix}^T$. 
Collocation Methods

- *Collocation methods* approximate $y$ by representing it in a basis

$$y(t) = v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t).$$

To construct equations, consider approximation for a set of collocation points $t_1, \ldots, t_n$ with $t_1 = a$ and $t_n = b$,

$$\forall i \in \{2, \ldots, n-1\} \quad v(t_i, x) = f(t_i, v(t_i, x)),$$

with two more equations typically obtained from boundary conditions at $t_1, t_n$.

- *Spectral methods* use polynomials or trigonometric functions for $\phi_i$, which are nonzero over most of $[a, b]$, while *finite element* methods leverage basis functions with local support (e.g. B-splines).

Eigenfunctions of differential operators are typically trigonometric functions or polynomials, hence the name “spectral methods”.
Solving BVPs by Optimization

We reformulate the collocation approximation as an optimization problem: Consider the simplified scenario \( f(t, y) = f(t) \) with residual equation,

\[
r(t, x) = v'(t, x) - f(t) = \sum_{j=1}^{n} x_j \phi'_j(t) - f(t)
\]

and minimize it using the objective function,

\[
F(x) = \frac{1}{2} \int_{a}^{b} \|r(t, x)\|_2^2 dt.
\]

The first-order optimality conditions of the optimization problem are a system of linear equations \( Ax = b \):

\[
0 = \frac{dF}{dx_i} = \int_{a}^{b} r(t, x)^T \frac{dr}{dx_i} dt = \int_{a}^{b} r(t, x)^T \phi'_i(t) dt
\]

\[
= \sum_{j=1}^{n} x_j \int_{a}^{b} \phi'_j(t)^T \phi'_i(t) dt - \int_{a}^{b} f(t)^T \phi'_i(t) dt
\]
Weighted Residual

- **Weighted residual methods** work by ensuring the residual is orthogonal with respect to a given set of weight functions:

  Rather than setting components of the gradient to zero, we instead have,

  \[
  \int_a^b r(t, x)^T w_i(t) dt = 0, \forall i \in \{1, \ldots, n\},
  \]

  which again yields a system of equations of the form \( Ax = b \) where

  \[
  a_{ij} = \int_a^b \phi_j'(t)^T w_i(t), \quad b_i = \int_a^b f(t)^T w_i(t).
  \]

  The collocation method is a weighted residual method where \( w_i(t) = \delta(t - t_i) \).

  - The Galerkin method is a weighted residual method where \( w_i = \phi_i \).

  Linear system with the stiffness matrix \( A \) and load vector \( b \) is

  \[
  0 = \sum_{j=1}^n x_j \underbrace{\int_a^b \phi_j'(t)^T \phi_i(t) dt}_a - \underbrace{\int_a^b f(t)^T \phi_i(t) dt}_b.
  \]
Linear BVPs by Optimization

Let's apply the Galerkin method to the more general linear ODE
\[ f(t, y) = A(t)y(t) + b(t) \]
with residual equation,

First, choose basis functions \( \{\phi_i\}_{i=1}^n \) to satisfy the boundary conditions, so solution automatically satisfies them, then minimize the residual,

\[ r = v' - Av - b, \quad \text{so that} \quad r(t, x) = \sum_{j=1}^{n} x_j (\phi'_j(t) - A(t)\phi_j(t)) - b(t). \]

The Galerkin method, minimizes the residual by orthogonality with respect to a set of test functions that is the same as the set of basis functions,

\[
0 = \int_a^b r(t, x)^T \phi_i(t) dt \\
= \sum_{j=1}^{n} x_j \int_a^b (\phi'_j(t) - A(t)\phi_j(t))^T \phi_i(t) dt - \int_a^b b(t)^T \phi_i(t) dt
\]
Nonlinear BVPs: Poisson Equation

In practice, BVPs are at least second order and it's advantageous to work in the natural set of variables.

Consider the Poisson equation \( u'' = f(t) \) with boundary conditions \( u(a) = u(b) = 0 \) and define a localized basis of hat functions:

\[
\phi_i(t) = \begin{cases} 
(t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\
(t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\
0 & : \text{otherwise}
\end{cases}
\]

where \( t_0 = t_1 = a \) and \( t_{n+1} = t_n = b \).

Trying to define the residual equation as usual, we obtain

\[
r = v'' - f, \text{ so that } r(t, x) = \sum_{j=1}^{n} x_j \phi_j''(t) - f(t).
\]

However, \( \phi_j''(t) \) is undefined, since \( \phi_j'(t) \) is discontinuous at \( t_{j-1}, t_j, t_{j+1} \).
The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in weak form:

For any solution $u$, if test function $\phi_i$ satisfies the boundary conditions, the ODE satisfies the weak form, 

$$
\int_a^b f(t)\phi_i(t)\,dt = \int_a^b u''(t)\phi_i(t)\,dt = u'(b)\phi_i(b) - u'(a)\phi_i(a) - \int_a^b u'(t)\phi_i'(t)\,dt
$$

$$
= - \int_a^b u'(t)\phi_i'(t)\,dt.
$$

Note that the final equation contains no second derivatives, and subsequently we can form the linear system $Ax = b$ with,

$$
a_{ij} = - \int_a^b \phi_j'(t)\phi_i'(t)\,dt, \quad b_i = \int_a^b f(t)\phi_i(t)\,dt.
$$

The finite element method thus searches the larger (once-differentiable) function space to find a solution $u$ that is in a (twice-differentiable) subspace.
Hat functions are linear instances of \textit{B-splines}:
\begin{quote}
\textit{B-splines of degree }$k$\textit{ are }$k$\textit{-times differentiable. For higher-order ODEs or high-order convergence with }$h$, \textit{its necessary to use }$k > 1$.
\end{quote}

Finite-element methods readily generalize to PDEs:
\begin{quote}
\textit{In its most basic form each element corresponds to a triangle (2D) or quadrilateral (3D).}
\end{quote}
A typical second-order scalar BVP eigenvalue problem has the form

\[ u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions} \quad u(a) = 0, u(b) = 0 \]

Let's first consider \( f(t, u, u') = g(t)u \), in which case we can approximate the solution at a set of points \( t_1, \ldots, t_n \) using finite differences,

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda g_i y_i, \]

which corresponds to a tridiagonal matrix eigenvalue problem \( Ay = \lambda y \) via

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{g_i h^2} = \lambda y_i. \]
Generalized eigenvalue problems arise from more sophisticated ODEs,

\[ u'' = \lambda (g(t)u + h(t)u'), \quad \text{with boundary conditions} \quad u(a) = 0, u(b) = 0 \]

We can approximate each of the derivatives at a set of points \( t_1, \ldots, t_n \) using finite differences,

\[
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda \left( g_i + \frac{y_{i+1} - y_{i-1}}{2h} \right) y_i.
\]

which can be expressed as a generalized matrix eigenvalue problem

\[ Ay = \lambda By \]

where both \( A \) and \( B \) are tridiagonal.