CS 450: Numerical Anlaysis

Lecture 26

Chapter 10 Boundary Value Problems for Ordinary Differential Equations

Numerical Methods for Boundary Value Problems

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Finite Difference Methods

- Lets derive the finite difference method for the ODE BVP defined by

$$u'' + 1000(1 + t^2)u = 0$$

with boundary conditions $u(-1) = 3$ and $u(1) = -3$.

Using a discretization with points $t_1, \ldots, t_n$, $t_{i+1} - t_i = h$, and a centered difference approximation for u'' we obtain

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + 1000(1 + t_i)u_i = 0.$$

We can rewrite the above using linear equations with matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & & & & \\ 1/h^2 & -2/h^2 & 1/h^2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/h^2 & -2/h^2 & 1/h^2 \\ & & & & 1 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{B} = \begin{bmatrix} 0 & & & \\ 0 & 1000(1+t_2) & & \\ & & \ddots & \\ & & 1000(1+t_{n-1}) & 0 \\ & & & 0 \end{bmatrix}$$

and solve the system $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \begin{bmatrix} 3 & 0 & \cdots 0 & -3 \end{bmatrix}^T$.

Collocation Methods

- *Collocation methods* approximate $\boldsymbol{y}$ by representing it in a basis

$$\boldsymbol{y}(t) = \boldsymbol{v}(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \boldsymbol{\phi}_i(t).$$

  To construct equations, consider approximation for a set of collocation points $t_1, \ldots, t_n$ with $t_1 = a$ and $t_n = b$,

$$\forall_{i \in \{2,\ldots,n-1\}} \quad \boldsymbol{v}(t_i, \boldsymbol{x}) = \boldsymbol{f}(t_i, \boldsymbol{v}(t_i, \boldsymbol{x})),$$

  with two more equations typically obtained from boundary conditions at t_1, t_n.

- *Spectral methods* use polynomials or trigonometric functions for ϕ_i, which are nonzero over most of $[a, b]$, while *finite element* methods leverage basis functions with local support (e.g. B-splines).

  Eigenfunctions of differential operators are typically trigonometric functions or polynomials, hence the name "spectral methods".

Solving BVPs by Optimization

- We reformulate the collocation approximation as an optimization problem: *Consider the simplified scenario $\boldsymbol{f}(t, y) = \boldsymbol{f}(t)$ with residual equation,*

$$\boldsymbol{r}(t, \boldsymbol{x}) = \boldsymbol{v}'(t, \boldsymbol{x}) - \boldsymbol{f}(t) = \sum_{j=1}^{n} x_j \boldsymbol{\phi}_j'(t) - \boldsymbol{f}(t)$$

and minimize it using the objective function,

$$F(\boldsymbol{x}) = \frac{1}{2} \int_a^b ||\boldsymbol{r}(t, \boldsymbol{x})||_2^2 dt.$$

- The first-order optimality conditions of the optimization problem are a system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{aligned} \mathbf{0} = \frac{dF}{dx_i} &= \int_a^b \boldsymbol{r}(t, \boldsymbol{x})^T \frac{d\boldsymbol{r}}{dx_i} dt = \int_a^b \boldsymbol{r}(t, \boldsymbol{x})^T \boldsymbol{\phi}_i'(t) dt \\ &= \sum_{j=1}^{n} x_j \underbrace{\int_a^b \boldsymbol{\phi}_j'(t)^T \boldsymbol{\phi}_i'(t) dt}_{a_{ij}} - \underbrace{\int_a^b \boldsymbol{f}(t)^T \boldsymbol{\phi}_i'(t) dt}_{b_i} \end{aligned}$$

Weighted Residual

- *Weighted residual methods* work by ensuring the residual is orthogonal with respect to a given set of weight functions:
  Rather than setting components of the gradient to zero, we instead have,
  $$\int_a^b \boldsymbol{r}(t,\boldsymbol{x})^T \boldsymbol{w}_i(t)dt = 0, \forall i \in \{1,\dots,n\},$$
  which again yields a system of equations of the form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *where*
  $$a_{ij} = \int_a^b \boldsymbol{\phi}_j'(t)^T \boldsymbol{w}_i(t), \quad b_i = \int_a^b \boldsymbol{f}(t)^T \boldsymbol{w}_i(t).$$
  The collocation method is a weighted residual method where $\boldsymbol{w}_i(t) = \boldsymbol{\delta}(t - t_i)$.
- The Galerkin method is a weighted residual method where $\boldsymbol{w}_i = \boldsymbol{\phi}_i$.
  Linear system with the stiffness matrix $\boldsymbol{A}$ *and load vector* $\boldsymbol{b}$ *is*
  $$\boldsymbol{0} = \sum_{j=1}^n x_j \underbrace{\int_a^b \boldsymbol{\phi}_j'(t)^T \boldsymbol{\phi}_i(t)dt}_{a_{ij}} - \underbrace{\int_a^b \boldsymbol{f}(t)^T \boldsymbol{\phi}_i(t)dt}_{b_i}.$$

Linear BVPs by Optimization

- Lets apply the Galerkin method to the more general linear ODE $\boldsymbol{f}(t, y) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ with residual equation,
  First, choose basis functions $\{\phi_i\}_{i=1}^n$ *to satisfy the boundary conditions, so solution automatically satisfies them, then minimize the residual,*

  $$\boldsymbol{r} = \boldsymbol{v}' - \boldsymbol{A}\boldsymbol{v} - \boldsymbol{b}, \ \text{ so that } \ \boldsymbol{r}(t, \boldsymbol{x}) = \sum_{j=1}^{n} x_j(\boldsymbol{\phi}_j'(t) - \boldsymbol{A}(t)\boldsymbol{\phi}_j(t)) - \boldsymbol{b}(t).$$

  The Galerkin method, minimizes the residual by orthogonality with respect to a set of test functions that is the same as the set of basis functions,

  $$\begin{aligned} \boldsymbol{0} &= \int_a^b \boldsymbol{r}(t, \boldsymbol{x})^T \boldsymbol{\phi}_i(t) dt \\ &= \sum_{j=1}^{n} x_j \int_a^b (\boldsymbol{\phi}_j'(t) - \boldsymbol{A}(t)\boldsymbol{\phi}_j(t))^T \boldsymbol{\phi}_i(t) dt - \int_a^b \boldsymbol{b}(t)^T \boldsymbol{\phi}_i(t) dt \end{aligned}$$

Nonlinear BVPs: Poisson Equation

In practice, BVPs are at least second order and its advantageous to work in the natural set of variables.

- Consider the Poisson equation $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:

$$\phi_i(t) = \begin{cases} (t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\ (t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\ 0 & : \text{otherwise} \end{cases}$$

  where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$.
  Trying to define the residual equation as usual, we obtain

$$r = v'' - f, \ \text{so that} \ r(t, \boldsymbol{x}) = \sum_{j=1}^{n} x_j \phi_j''(t) - f(t).$$

  However, $\phi_j''(t)$ *is undefined, since* $\phi_j'(t)$ *is discontinuous at* t_{j-1}, t_j, t_{j+1}.

Weak Form and the Finite Element Method

- The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in *weak form*:
  For any solution u, if test function ϕ_i satisfies the boundary conditions, the ODE satisfies the weak form,

$$\int_a^b f(t)\phi_i(t)dt = \int_a^b u''(t)\phi_i(t)dt = u'(b)\underbrace{\phi_i(b)}_{0} - u'(a)\underbrace{\phi_i(a)}_{0} - \int_a^b u'(t)\phi_i'(t)dt$$

$$= -\int_a^b u'(t)\phi_i'(t)dt.$$

  Note that the final equation contains no second derivatives, and subsequently we can form the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ with,

$$a_{ij} = -\int_a^b \phi_j'(t)\phi_i'(t)dt, \quad b_i = \int_a^b f(t)\phi_i(t)dt.$$

  The finite element method thus searches the larger (once-differentiable) function space to find a solution u that is in a (twice-differentiable) subspace.

Finite Element Methods in Practice

- Hat functions are linear instances of *B-splines*:

  B-splines of degree k are k-times differentiable. For higher-order ODEs or high-order convergence with h, its necessary to use $k > 1$.

- Finite-element methods readily generalize to PDEs:

  In its most basic form each element corresponds to a triangle (2D) or quadrilateral (3D).

Eigenvalue Problems with ODEs

- A typical second-order scalar BVP eigenvalue problem has the form

$$u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0$$

Lets first consider $f(t, u, u') = g(t)u$*, in which case we can approximate the solution at a set of points* $t_1, \dots, t_n$ *using finite differences,*

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda g_i y_i,$$

which corresponds to a tridiagonal matrix eigenvalue problem $\boldsymbol{A}\boldsymbol{y} = \lambda \boldsymbol{y}$ *via*

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{g_i h^2} = \lambda y_i.$$

Eigenvalue Problems with ODEs

- Generalized eigenvalue problems arise from more sophisticated ODEs,

$$u'' = \lambda(g(t)u + h(t)u'), \quad \text{with boundary conditions } \; u(a) = 0, u(b) = 0$$

We can approximate each of the derivatives at a set of points $t_1, \dots, t_n$ using finite differences,

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda\left(g_i + \frac{y_{i+1} - y_{i-1}}{2h}\right)y_i.$$

which can be expressed as a generalized matrix eigenvalue problem

$$\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{B}\boldsymbol{y}$$

where both $\boldsymbol{A}$ and $\boldsymbol{B}$ are tridiagonal.