# CS 450: Numerical Anlaysis

## Lecture 27

## Chapter 11 Partial Differential Equations

### Numerical Methods for Partial Differential Equations

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# Partial Differential Equations

- Partial differential equations (PDEs) are equations describe physical laws and other continuous phenomena:

  partial derivatives in multiple variables

  - electromagnetics
  - fluid flow
  - quantum physics

- A simple PDE is the advection equation, which describes basic phenomena in fluid flow:

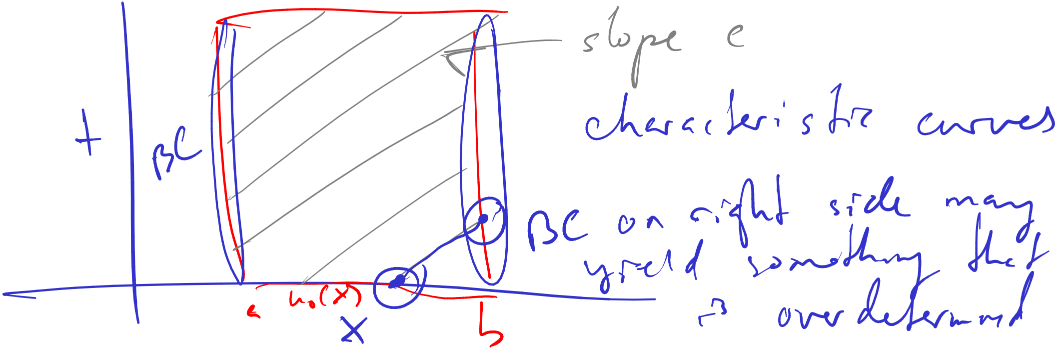
$$u_t = -a(t, x)u_x \qquad t \geq 0$$

Cauchy

I.C. $u(0, x) = u_0(x) \quad x \in [a, b]$

$u_t = -c u_x$

can have further BC



slope c
characteristic curves
BC on right side may yield something that
→ overdetermined
t
BC
x
a
b
$u_0(x)$

# Properties of PDEs

- A *characteristic* of a PDE is a level curve in the solution:

$x(t)$ = characteristic curve

$u(t, x(t)) = c$ — constant

$\frac{\partial x}{\partial t}(t) = a(t, x(t))$ ← advection equation

yields ODE
$x(0) = x_0$
$x'(t) = a(t, x(t))$

- The order of a PDE is the highest-order of any partial derivative appearing in the PDE:

advection equation is first order.

second order implies that we have e.g.

$u_{xx} = \frac{\partial^2 u}{\partial x \partial x}$ $\quad u_{tt} \quad u_{xy}$

# Types of PDEs

- Some of the most important PDEs are second order:

Heat equation (diffusion) $u_t = u_{xx}$

Wave equation (oscillation) $u_{tt} = u_{xx}$

Laplace equation (steady state) $u_{xx} + u_{yy} = 0$

IVPs

BVP

- The *discriminant* determines the canonical form of second-order PDEs:

Constant-coefficient linear second-order PDE

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0$$

$$r = b^2 - 4ac \begin{cases} r > 0 : & \text{hyperbolic (wave equation)} \\ r = 0 : & \text{parabolic (heat equation)} \\ r < 0 : & \text{elliptic (Laplace equation)} \end{cases}$$

# Method of Lines

- Semidiscrete methods obtain an approximation to the PDE by solving a system of ODEs, e.g. consider heat equation

$$u_t = cu_{xx} \text{ on } 0 \leq x \leq 1, \quad u(0,x) = f(x), u(t,0) = u(t,1) = 0$$

$x_1, \dots, x_n \in [0,1]$ $\quad x_i - x_{i-1} = \Delta x$

$$u_{xx}(t, x_i) \approx \frac{u(t,x_{i+1}) - 2u(t,x_i) + u(t,x_{i-1})}{(\Delta x)^2}$$

$y_i(t) = u(t, x_i)$. $\quad y_i'(t) = c\,\dfrac{y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)}{(\Delta x)^2}$

- This *method of lines* often yields a stiff ODE:

rapidly + slowly varying components

$y'(t) = Ay(t)$

eigenvalues of $A$ will range from $0$ to $-4c/(\Delta x)^2$

# Semidiscrete Collocation

- Instead of finite-differences, we can express $u(t,x)$ in a spatial basis:
- For the heat equation $u_t = cu_{xx}$, we obtain an ODE:

# Fully Discrete Methods

- Generally, both time and space dimensions are discretized, for example using finite differences:

# Implicit Fully Discrete Methods

- When using Euler's method for the heat equation, to stay in stability region, require

$$\Delta t = O\Big((\Delta x)^2\Big)$$

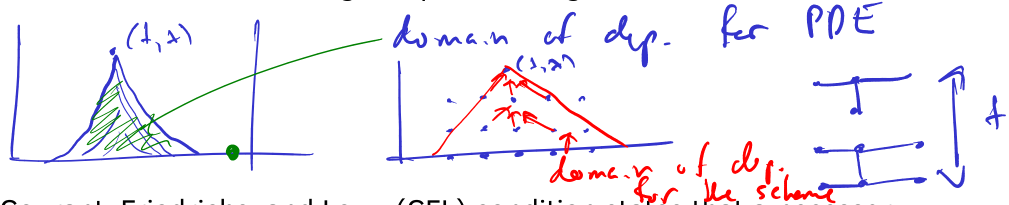
# Convergence and Stability

- *Lax Equivalence Theorem*: consistency + stability = convergence

- Stability can be ascertained by spectral or Fourier analysis:

# CFL Condition

- The domain of dependence of a PDE for a given point $(t, x)$ is the portion of the problem domain influencing this point through the PDE:




- The Courant, Friedrichs, and Lewy (CFL) condition states that a *necessary* condition for an explicit finite-differencing scheme to be stable for a hyperbolic PDE is that the domain of the dependence of the PDE be contained in the domain of dependence of the scheme:

# Time-Independent PDEs

- We now turn our focus to time-independent PDEs as exemplified by the *Helmholtz equation*:

$$u_{xx} + u_{yy} + \lambda u = f(x, y)$$

- We discretize as before, but no longer perform time stepping:

# Finite-Differencing for Poisson

- Consider the Poisson equation with equispaced mesh-points on $[0, 1]$:

# Multidimensional Finite Elements

- There are many ways to define localized basis functions, for example in the 2D FEM method[1]: