CS450-Recitation-2

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Overview

Content

1. Recap of Linear Algebra
2. LU/Gaussian Elimination
3. Practice Questions (In the mean time)

System of Equations

$$2x - y = 1$$
$$x + y = 2$$

The intersection of the two lines gives the unique point $(x, y) = (1, 1)$, which is the solution. However, if two lines are parallel, cases need to be discussed separately.

$$x \begin{bmatrix} 2 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

System of Equations

Matrix form:

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus, $\mathbf{Ax} = \mathbf{b}$. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} \end{bmatrix}$$

$\mathbf{Ax} = x\mathbf{a_1} + y\mathbf{a_2}$
Linear combination of columns.

System of Equations

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 \end{bmatrix}$$

System of Equations

Consider $\mathbf{Ax} = \mathbf{b}$:

- Existence and uniqueness of solution depend on whether $\mathbf{A}$ is singular or nonsingular.
- Previous case, nonsingular if two lines intersect.
- In singular cases, infinite solutions if $\mathbf{b} \in span(\mathbf{A})$ else none.
- $\mathbf{A}$ is nonsingular iff $\mathbf{A}^{-1}$ exits.
- $\mathbf{A}$ is nonsingular iff $det(\mathbf{A}) \neq 0$.
- Check Michael/Lecture notes for more equivalent definition of nonsingular

Practice!

$$\mathbf{A} = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}$$

- What is the determinant of **A**?
- In floating-point arithmetic, for what range of values of ϵ will **A** be singular?
- In floating-point arithmetic, for what range of values of ϵ will the computed value of the determinant be zero?

Practice!

Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is singular.

Solve the system

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Easy to solve if **A** is orthogonal, diagonal.
Easy to solve if **A** is upper triangular(backward-substitution) or lower triangular(Forward-substitution).

Practice!

Within this linear system, it's easy to achieve triangularity.

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solve the system

Within this linear system, it's easy to achieve.

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

More details will be covered in the following lectures.

System of Equations

Linear Algebra Recap:

Given an $m \times n$ matrix $\mathbf{A}$,

Definition 1

the rank of a matrix $\mathbf{A}$ is defined as the dimension of the vector space generated (or spanned) by its columns. This corresponds to the maximal number of linearly independent columns of $\mathbf{A}$.

Definition 2

the null-sapce of a matrix $\mathbf{A}$ is defined as is the set of solutions to the equation $\mathbf{Ax} = \mathbf{0}$. The dimension of the null space of $\mathbf{A}$ is called the nullity of $\mathbf{A}$.

Rank Nullity theorem

Given an $m \times n$ matrix $\mathbf{A}$, $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$.

Practice!

$$\mathbf{A} = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}$$

- What is the LU factorization of **A**?
- In floating-point arithmetic, for what range of values of ϵ will the computed value of U be singular?

Practice!

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the unit vector spanning the null space of $\mathbf{A}$.

Thanks