CS 450: Numerical Anlaysis

Lecture 25
Chapter 10 Boundary Value Problems for Ordinary Differential Equations
Conditioning and Solutions to Boundary Value Problems

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Boundary Value Problems for ODEs

- Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary. Consider a first order ODE $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y})$ with general *linear boundary conditions*:

$$\boldsymbol{B}_a \boldsymbol{y}(a) + \boldsymbol{B}_b \boldsymbol{y}(b) = \boldsymbol{c}$$

  - *for IVP, simple case of Dirichlet (value) condition $\boldsymbol{B}_a = \boldsymbol{I}$, $\boldsymbol{B}\boldsymbol{B}_b = \boldsymbol{0}$*
  - *conditions are separate if $\boldsymbol{B}_a \neq \boldsymbol{0}$ and $\boldsymbol{B}_b \neq \boldsymbol{0}$*

- High-order boundary conditions can be reduced to first-order like ODEs themselves:

  The Neumann boundary condition gives $\boldsymbol{y}'(a)$, which simply implies we have $\boldsymbol{u}_2(a) = \boldsymbol{y}'(a)$ if we use $\boldsymbol{u}_1 = \boldsymbol{y}$ and $\boldsymbol{u}_2 = \boldsymbol{y}'$.

Boundary Value Problems for ODEs

- Can derive the solutions to a linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ from solutions to homogeneous linear ODE $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}(t)$ IVPs:

  Let the solution to the homogeneous ODE IVP with $\boldsymbol{y}_i(a) = \boldsymbol{e}_i$ be given by the ith column of $\boldsymbol{Y}(t)$, so

  $$\boldsymbol{Y}(t) = \boldsymbol{I} + \int_a^t \boldsymbol{A}(s)\boldsymbol{Y}(s)ds.$$

  Now, we seek to derive a solution of the form $\boldsymbol{y}(t) = \boldsymbol{Y}(t)\boldsymbol{u}(t)$ with $\boldsymbol{u}(0) = \boldsymbol{y}(a)$, which satisfies the given boundary condition,

  $$\boldsymbol{B}_a\boldsymbol{Y}(a)\boldsymbol{y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b)\boldsymbol{y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b)\int_a^b \boldsymbol{u}'(s)ds = \boldsymbol{c}$$

  $$\underbrace{(\boldsymbol{B}_a\boldsymbol{Y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b))}_{\boldsymbol{Q}}\boldsymbol{y}(a) = \boldsymbol{c} - \boldsymbol{B}_b\boldsymbol{Y}(b)\int_a^b \boldsymbol{u}'(s)ds,$$

  the existence of solution $\boldsymbol{y}$, thus generally depends on invertibility of $\boldsymbol{Q}$.

Boundary Value Problems for ODEs

- Can derive the solutions to a linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ from solutions to homogeneous linear ODE $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}(t)$ IVPs:

To determine $\boldsymbol{u}(t) = \boldsymbol{y}(a) + \int_a^t \boldsymbol{u}'(s)ds$, we use the differential equation

$$\boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t) = \boldsymbol{y}'(t) = \boldsymbol{Y}'(t)\boldsymbol{u}(t) + \boldsymbol{Y}(t)\boldsymbol{u}'(t)$$
$$= \boldsymbol{A}(t)\underbrace{\boldsymbol{Y}(t)\boldsymbol{u}(t)}_{\boldsymbol{y}(t)} + \boldsymbol{Y}(t)\boldsymbol{u}'(t),$$
$$\boldsymbol{u}'(t) = \boldsymbol{Y}^{-1}(t)\boldsymbol{b}(t).$$

Thus, given $\boldsymbol{u}(t) = \boldsymbol{y}(a) + \int_a^t \boldsymbol{Y}^{-1}(s)\boldsymbol{b}(s)ds$, the overall solution is

$$\boldsymbol{y}(t) = \boldsymbol{Y}(t)\boldsymbol{u}(t)$$
$$= \underbrace{\boldsymbol{Y}(t)\boldsymbol{Q}^{-1}\left(\boldsymbol{c} - \boldsymbol{B}_b\boldsymbol{Y}(b)\int_a^b \underbrace{\boldsymbol{Y}^{-1}(s)\boldsymbol{b}(s)}_{\boldsymbol{u}'(s)}\,ds\right)}_{\boldsymbol{y}(a)} + \boldsymbol{Y}(t)\int_a^t \underbrace{\boldsymbol{Y}^{-1}(s)\boldsymbol{b}(s)}_{\boldsymbol{u}'(s)}\,ds$$

Linear ODE BVP Green's Function

▶ We now express our solution (with form $y(t) = Y(t)(u(a) + \int_a^t u'(s)ds)$) in the form $y(t) = s(t) + \int_a^b G(t,s)b(s)ds$ where G is the *Green's function*: *Using the fact that,*

$$
Y(t)Q^{-1}\underbrace{(B_a Y(a) + B_b Y(b))}_{Q}\int_a^t Y^{-1}(s)b(s)ds = Y(t)\int_a^t Y^{-1}(s)b(s)ds,
$$

we can express our previous solution in the form,

$$
y(t) = Y(t)Q^{-1}\left(c + B_a Y(a)\int_a^t Y^{-1}(s)b(s)ds - B_b Y(b)\int_t^b Y^{-1}(s)b(s)ds\right),
$$

which allows us to derive the Green's function,

$$
G(t,s) = Y(t)Q^{-1}I(s)Y^{-1}(s), \quad I(s) = \begin{cases} B_a Y(a) & : s < t \\ B_b Y(b) & : s \geq t \end{cases}
$$

Conditioning of Linear ODE BVPs

- For any given $b(t)$ and c, the solution to the BVP can be written in the form:

$$y(t) = \Phi(t)c + \int_a^b G(t, s)b(s)ds$$

  $\Phi(t) = Y(t)Q^{-1}$ *is the fundamental matrix, which like the Green's function is associated with the homogeneous ODE as well as its linear boundary condition matrices B_a and B_b, but is independent $b(t)$ and c.*

- The absolute condition number of the BVP is $\kappa = \max\{||\Phi||_\infty, ||G||_\infty\}$:

  This sensitivity measure enables us to bound the perturbation $||\hat{y} - y||_\infty$ with respect to the magnitude of a perturbation to $b(t)$ or c.

Shooting Method for ODE BVPs

▶ For linear ODEs, we constructed solutions from IVP solutions in $\boldsymbol{Y}(t)$, which suggests a method for solving BVPs by reduction to IVPs:

The shooting method iteratively (for $k = 1, 2, \ldots$) constructs approximate initial value guesses $\hat{\boldsymbol{y}}^{(k)}(a) \approx \boldsymbol{y}(a)$, solves the resulting IVP, and checks the quality of the solution at the new boundary,

$$||\boldsymbol{B}_b \hat{\boldsymbol{y}}^{(k)}(b) - \boldsymbol{B}_a \hat{\boldsymbol{y}}^{(k)}(a) - \boldsymbol{c}||,$$

the initial conditions for the next shot, $\hat{\boldsymbol{y}}^{(k+1)}(a)$ can be constructed by a 1D root finding technique on

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{B}_a \boldsymbol{x} + \boldsymbol{B}_b \boldsymbol{y}_{\boldsymbol{x}}(b) - \boldsymbol{c}, \text{ where } \boldsymbol{y}_{\boldsymbol{x}}(b) \text{ is the IVP Solution with } \boldsymbol{y}_{\boldsymbol{x}}(a) = \boldsymbol{x}.$$

▶ *Multiple shooting* employs the shooting method over subdomains:

Conditioning of shooting method depends on stability of IVPs, which can be worse than conditioning of the BVP. However, the shooting problems are interdependent, as they must satisfy continuity conditions on boundaries between them, leading to a system of nonlinear equations.

Finite Difference Methods

- ▶ Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

  Finite difference methods works by obtaining a solution on points $t_1, \ldots, t_n$, so that $\hat{\boldsymbol{y}}_k \approx \boldsymbol{y}(t_k)$ by finite-difference formulae, for example

  $$\boldsymbol{f}(t, \boldsymbol{y}) = \boldsymbol{y}'(t) \approx \frac{\boldsymbol{y}(t+h) - \boldsymbol{y}(t-h)}{2h} \Rightarrow \boldsymbol{f}(t_k, \hat{\boldsymbol{y}}_k) = \frac{\hat{\boldsymbol{y}}_{k+1} - \hat{\boldsymbol{y}}_{k-1}}{t_{k+1} - t_{k-1}}$$

  the resulting system of equations can be solved by standard methods and is linear if $\boldsymbol{f}$ is linear.

- ▶ Convergence to solution is obtained with decreasing step size h so long as the method is consistent and stable:

  Consistency implies that the truncation error goes to zero, while stability ensures input perturbations have bounded effect on solution.