$$f(\vec{x}) = \vec{0}$$ "smooth"

Today:

Top 10

nonlinear
- ↳ rates of convergence
- ↳ bisection
- ↳ Newton (1D)
- ↳ multiple dim

Rates of Convergence

What is *linear convergence*? *quadratic convergence*?

$\vec{x}_k \to \vec{x}^*$ $\quad \vec{e}_k = \vec{x}_k - \vec{x}^*$

NOT QUITE: $\|e_{k+1}\| \le C \cdot \|e_k\|^r$ ← rate of convergence

rate of conv. 1 (linear): "constant number of accurate digits" per it.

rate of conv. 2 (quadratic)

ideally: $0 \le C < 1$

"linearly convergent": $\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C$ $\quad >0, <\infty$

About Convergence Rates

Demo: Rates of Convergence

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

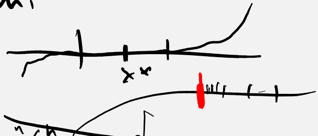
Stopping Criteria

Comment on the 'foolproof-ness' of these stopping criteria:

1. $|f(x)| < \varepsilon$ ('residual is small')
2. $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \varepsilon$
3. $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \,/\, \|\mathbf{x}_k\| < \varepsilon$

Fail when:

1.



2. "ε" small?

might stall far away

3. what if $x^* = 0$?

Bisection Method

$$\|e_{k+1}\| \le C \cdot \|e_k\|$$

Demo: Bisection Method

What's the rate of convergence? What's the constant?

linear $\quad\quad$ $1/2$

Fixed Point Iteration

$$\begin{aligned} x_0 &= \langle \text{starting guess} \rangle \\ x_{k+1} &= g(x_k) \end{aligned}$$

Demo: Fixed point iteration

When does fixed point iteration converge? Assume g is smooth.

Fixed Point Iteration: Convergence cont'd.

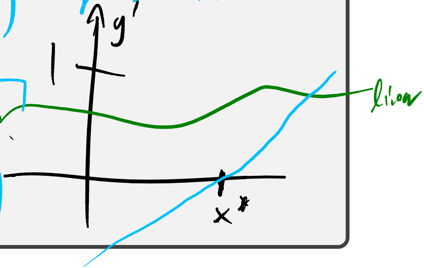
Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

$$e_{k+1} = g'(\theta_k)\, e_k \approx g''(x^*)\, \underbrace{e_k \cdot e_k}_{e_k^2}$$

- linearly conv. (in a nbhood) if $|g'(x^*)| < 1$

  quadratic conv. (in a nbhood) if $|g'(x^*)| = 0$

$$g'(x_k) \approx g''(x^*) \cdot \underbrace{(x_k - x^*)}_{e_k}$$



Newton's Method

Derive Newton's method.

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

Drawbacks of Newton?

Demo: Newton's method
Demo: Convergence of Newton's Method