Announcements
- HW0
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Today
- Errors
- Conditioning
- FP
Forward/Backward Error

Suppose \textit{want} to compute \( y = f(x) \), but \textit{approximate} \( \hat{y} = \hat{f}(x) \).

What are the forward error and the backward error?

\[
\text{Forward error:} \quad |\Delta y| = |f(x) - \hat{f}(x)| \quad \left\{ \begin{array}{l}
\text{abs} \\
\text{rel}
\end{array} \right.
\]

\[
\frac{|\Delta y|}{|y|} \quad \left\{ \begin{array}{l}
\text{rel}
\end{array} \right.
\]

\[
\text{Backward error:} \quad f(x) = y \quad \left\{ \begin{array}{l}
\hat{f}(x) = \hat{y}
\end{array} \right.
\]

\[
\text{backward error}
\]
Forward/Backward Error: Example

Suppose you wanted \( y = \sqrt{2} \) and got \( \hat{y} = 1.4 \).
What’s the (magnitude of) the forward error?

\[ \Delta y = y - \hat{y} = 0.014 \ldots \]

relative forward error

\[ \frac{0.01}{0.01} = 1 \]

\[ \log_{10} 0.01 = -2 \]

0.000 1234567000
0.000 1234
Forward/Backward Error: Example

Suppose you wanted \( y = \sqrt{2} \) and got \( \hat{y} = 1.4 \).
What’s the (magnitude of) the backward error?

\[
\sqrt{x} = y, \\
\sqrt{\hat{x}} = 1.4, \\
x^2 = 1.4 \approx 1.96, \\
x - \hat{x} = 0.04 \leftarrow \text{abs. backward error}
\]

\[
\Rightarrow \left| \frac{x - \hat{x}}{x} \right| = 0.02
\]
What do you observe about the relative magnitude of the relative errors?

\[ \text{rel. bw} \quad 0.02 \quad \text{rel. fw} \quad 0.01 \ldots \]
Sensitivity and Conditioning

What can we say about amplification of error?

\[ |\text{rel. bw. error}| \leq k \leq |\text{rel. bw. error}| \]

\[ \text{cond} = k = \max_x \frac{|\Delta y|/|y|}{|\Delta x|/|x|} \]

smallest number that does this:

condition number
\[ \frac{\Delta x^1}{|x|} \leq 10^{-6} \]

\[ \kappa = 10,000 \approx 10^4 \]

\[ \frac{\Delta y^1}{|y|} \leq 10^{-2} \]
Example: Condition Number of Evaluating a Function

\[ \Delta y = f(x + \Delta x) - f(x) \approx f'(x) \cdot \Delta x \]

\[ y = f(x) \]. Assume \( f \) differentiable.

\textbf{Demo: Conditioning of Evaluating tan}
\[
\frac{|x| \cos x}{|\sin x|} \leq \frac{|x|}{|dx|} \leq \frac{1}{2}
\]
Stability and Accuracy

When is a method **accurate**?

Output is close to the true answer.

When is a method **stable**?

"Backward stable"

A method is (back) stable if the result it produces is the exact solution to a nearby problem.
Getting into Trouble with Accuracy and Stability

\[ \beta(x) = 5 \]

How can I produce inaccurate results?

- Apply an inaccurate method
- Apply an unstable method to a well-conditioned problem
- Apply any type of method to an ill-conditioned input
In-Class Activity: Forward/Backward Error

In-class activity: Forward/Backward Error
Wanted: Real Numbers... in a computer

Computers can represent integers, using bits:

\[ 23 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = (10111)_2 \]

How would we represent fractions?
Fixed-Point Numbers

Suppose we use units of 64 bits, with 32 bits for exponents $\geq 0$ and 32 bits for exponents $< 0$. What numbers can we represent?

How many ‘digits’ of relative accuracy (think relative rounding error) are available for the smallest vs. the largest number?