Today

Announcements

- HW5
- Modular/testable numerics code
- Complaint period
- You may notice researchers from a psychology lab here on campus that are visiting our classroom today to make observations for a research study. We are going to continue with our class as usual while they observe.

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\mathbf{x} \neq \mathbf{0}$ is called an *eigenvector* of A if there exists a λ so that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

- In that case, λ is called an *eigenvalue*.
- The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- The *spectral radius* is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\}$$

Finding Eigenvalues

How do you find eigenvalues?

$$A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$$
$$\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \geqslant 5$ is no general formula for roots of polynomial. IOW: no.

- For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *approximate*.

Demo: Rounding in characteristic polynomial using SymPy

Multiplicity

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- *Algebraic Multiplicity*: multiplicity of the root of the characteristic polynomial $CP = (1-\lambda)^5 \cdots$
- *Geometric Multiplicity*: #of lin. indep. eigenvectors

In general: AM $\geqslant$ GM.
If AM $>$ GM, the matrix is called *defective*.

An Example

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}. \qquad \lambda 1 - A$$

$CP = (\lambda - 1)^2 \to$ ev. 1 w/ AM 2

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{matrix} x + y = x \\ y = 0 \end{matrix}$$

$$\hookrightarrow \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Diagonalizability

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A x_i = \lambda_i x_i$$

When is a matrix called *diagonalizable*?

Suppose I have n lin. indep. eigenvectors $(x_i)_{i=1}^n$

$$X = \begin{pmatrix} | & | & | & \\ x_1 & x_2 & x_3 & \\ | & | & | & \end{pmatrix}$$

$$AX = \begin{pmatrix} | & | & \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots \\ | & | & \end{pmatrix} = XD$$

$$X^{-1} A X = D$$

Similar Matrices

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose $Av = \lambda v$. $B = X^{-1}AX$. $w := X^{-1}v$

$$Bw = X^{-1}A\cancel{XX^{-1}}v = X^{-1}Av = \lambda X^{-1}v$$
$$= \lambda w.$$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ do?

Shift. $A \to A - \sigma I$

$$(A-\sigma I)x = Ax - \sigma I x = \lambda x - \sigma x = (\lambda - \sigma)x$$

Inversion. $A \to A^{-1}$

$$Ax = \lambda x \quad | \, A^{-1}\cdot$$

$$\Leftrightarrow x = \lambda A^{-1}x \quad \Leftrightarrow \quad \frac{1}{\lambda}x = A^{-1}x$$

Power. $A \to A^k$

$$A^k x = \lambda^k x$$

Eigenvalue Transformations (II)

Polynomial $A \to aA^2 + bA + cI$

$$(aA^2 + bA + cI)x = (a\lambda^2 + b\lambda + c)x$$

Similarity $T^{-1}AT$ with T invertible

↳ eigvalues stay same
↳ eigenvectors change

Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

Schur Form: Comments and Eigenvalues

For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag.

Also, if we knew how to compute Schur form, how would it help us find eigenvalues?

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \to A+E$.
What happens to the eigenvalues?

$$X^{-1}(A+E)X = D+F$$

$$(D+F)v = \mu v \longrightarrow \text{corresponds to } \mu \text{ eigenvalue of } A+E$$

$$\iff Fv = (\mu I - D)v$$

; assume μ doesn't match entries of D $\Rightarrow (\mu I - D)$ inv.

$$\iff (\mu I - D)^{-1} F v = v$$

$$\Rightarrow \cancel{\|v\|} = \|(\mu I - D)^{-1} F v\| \le \|(\mu I - D)^{-1}\| \, \|F\| \, \cancel{\|v\|}$$

$$\frac{1}{\|(\mu I - D)^{-1}\|} \le \|F\|$$

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} - 3I = \begin{pmatrix} -2 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

D

Sensitivity (II)

Suppose λ_k is the eigenvalue closest to μ

$X^{-1}(A+E)X = D + F$. Have $\|(\mu I - D)^{-1}\|^{-1} \leqslant \|F\|$.

$$\|(\mu I - D)^{-1}\|^{-1} = |\mu - \lambda_k|$$

$$|\mu - \lambda_k| = \|(\mu I - D)^{-1}\|^{-1} \le \|F\| = \|X^{-1}EX\|$$

$$\le \|X\|\|X^{-1}\| \|E\|$$

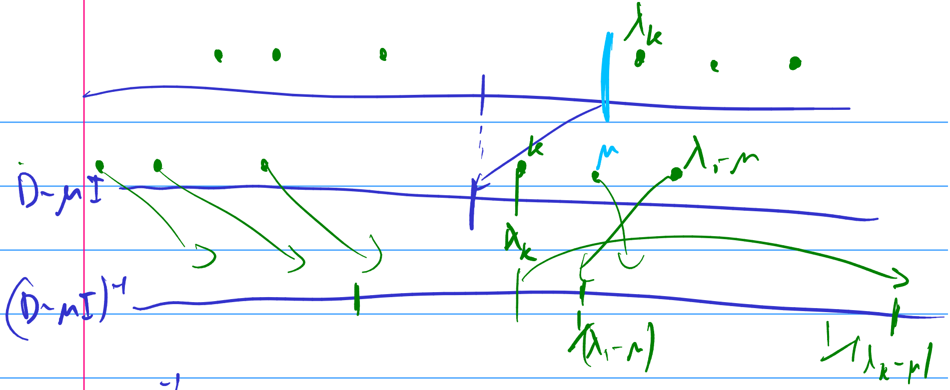
$$= \operatorname{cond}(X) \|E\|$$

how much has λ_k moved

$F = X^{-1}EX$

A symm. $\Rightarrow$ X orth.

$\hookrightarrow$ "The symm. eigenvalue problem is well-behaved"



$$\|(D-\mu I)^{-1}\|^{-1} = \frac{1}{|\lambda_k - \mu|}$$

$$\|(D-\mu I)^{-1}\|^{-1} = |\lambda_k - \mu|$$

Power Iteration

What are the eigenvalues of A^{1000}?

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$.
Further assume $\|\mathbf{x}_i\| = 1$.

$$x = \alpha x_1 + \beta x_2 + \cdots$$

$$A^{1000} x = A^{1000} \alpha x_1 + A^{1000} \beta x_2 + \cdots$$

$$= \alpha \lambda_1^{1000} x_1 + \beta \lambda_2^{1000} x_2 + \cdots$$

$$\frac{A^{1000} x}{\lambda_1^{1000}} = \alpha x_1 + \underbrace{\beta \Big(\underbrace{\frac{\lambda_2}{\lambda_1}}_{<1}\Big)^{1000}} + \cdots$$

super-brittle:

- $|\lambda_1| = |\lambda_2|$ breaks it
- overflow

- non-diagonalizable

Power Iteration: Issues?

What could go wrong with Power Iteration?

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?