

Today:

- Krylov
- Nonlinear

Announcements:

- Examlet 2
- HW7
- Final exam
- Class Transcribe

## Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$$\text{span}(x, Ax, A^2x, \dots) = \text{span}(Q_k)$$

$$AQ_n = Q_n H \longrightarrow$$

$$h_{jk} = q_j^T A q_k$$

$$Q_n^T A Q_n = \begin{array}{|c|} \hline \text{H} \\ \hline \end{array}$$

$$Q_k^T A Q_k = \begin{array}{|c|} \hline \text{H} \\ \hline \end{array}$$



Ritz values

Demo: Arnoldi Iteration (Part 1)

For sym: Lanczos iteration



Dense	LR
LR	Dense



## Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?



[Demo: Arnoldi Iteration](#) (Part 2)

## Computing the SVD (Kiddy Version)

How can I compute an SVD of a matrix  $A$ ?

← square

$$A^T A v_i = \lambda_i v_i$$

$$V^T A^T A V = \Sigma^2$$

↑  
diag

$$V \text{ (matrix of eigvec)} = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$$

orth. because  $A^T A$  sym.

$$\sigma_i = \sqrt{\lambda_i}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$\leadsto U = A V \Sigma^{-1}$$

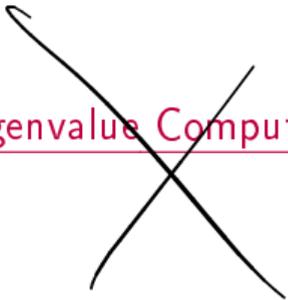
$$U^T U = \Sigma^{-1} \underbrace{V^T A^T A V}_{\Sigma^2} \Sigma^{-1} = \Sigma^{-1} \Sigma^2 \Sigma^{-1} = I$$

Demo: Computing the SVD

R-SVD

## In-Class Activity: Eigenvalue Computations

In-class activity: Eigenvalue Computations



# Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems

**Nonlinear Equations**

Iterative Procedures

Methods in One Dimension

Methods in  $n$  Dimensions ("Systems of Equations")

Optimization

Interpolation

Numerical Integration and Differentiation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

## Solving Nonlinear Equations

What is the goal here?

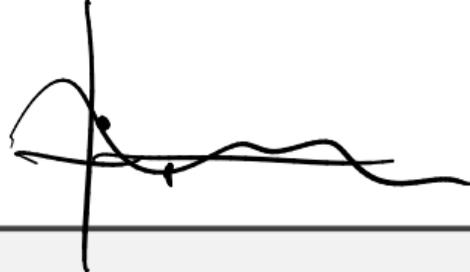
all equation solving can be written like this

$$f(\vec{x}) = \vec{0}$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

$$g(\vec{x}) = \vec{y} \rightsquigarrow f(x) = g(x) - \vec{y}$$

## Showing Existence



How can we show existence of a root?

1D - Intermediate value thm.

nD - Inverse function thm

$(f^{-1})' = \frac{1}{f'}$   
(guarantees existence of an inverse locally)

$f^{-1}$  helps because  
 $x = f^{-1}(0)$

## Contraction mapping

A function  $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called  
contractive if there is  $0 \leq \gamma < 1$  so that

$$\|g(x) - g(y)\| \leq \gamma \|x - y\|.$$

On a closed set  $S \subseteq \mathbb{R}^2$  with  $g(S) \subseteq S$ ,  
there exists a unique fixed point.

(horrible)  
example

$$g(x) := f(x) + x$$

## Sensitivity and Multiplicity

What is the sensitivity/conditioning of root finding?

$$\text{cond}(\text{root finding}) = \text{cond}(\text{evaluating the inverse function})$$

What are multiple roots?

$$f(x) = (x-15)^5$$

multiplicity  $m$

$$f(x) = 0$$
$$f'(x) = 0, \dots, f^{(m-1)}(x) \neq 0$$

How do multiple roots interact with conditioning?

multiple roots  $\Rightarrow$  terrible conditioning of root finding

# Rates of Convergence

What is *linear convergence*? *quadratic convergence*?

$$\vec{u}_k \quad \text{true solution: } \vec{u}$$
$$\vec{e}_k = \vec{u}_k - \vec{u}$$

An iterative method converges with rate  $r$  if

$$\lim_{k \rightarrow \infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} > 0 \\ < \infty \end{cases}$$

$$\|e_{k+1}\| \leq C \cdot \|e_k\|^2 \leftarrow \text{quadratic}$$

$$\text{most useful if } 0 < r < 1, \quad \|e_{k+1}\| \leq C \|e_k\| \leftarrow \text{linear}$$

Power iteration: linearly conv.

$$\|x_{k+1} - x\| \leq \frac{\lambda_2}{\lambda_1} \cdot \|x_k - x\|$$

## About Convergence Rates

### Demo: Rates of Convergence

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

