Today:

- Krylov
- Nonlinear

Announcements:

- Exam let 2
- HW7
- Final exam
- ClassTranscribe

Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$$\operatorname{span}(x, Ax, A^2x, \dots) = \operatorname{span}(Q_k)$$

$$AQ_n = Q_n H \longrightarrow h_{jk} = q_j^T A q_k$$

$$Q_n^T A Q_n =$$

$$Q_k^T A Q_k =$$

Ritz values

Demo: Arnoldi Iteration (Part 1)

For sym: Lanczos iteration

$$\left(\; Ax \; \right)$$

S $\quad 1$

Sx

Dense	LR
LR	Dense

Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

Demo: Arnoldi Iteration (Part 2)

Computing the SVD (Kiddy Version)

How can I compute an SVD of a matrix A? square

$A^T A v_i = \lambda_i v_i$

$V^T A^T A V = \Sigma^2$

diag

V (matrix of eigvec) $= \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

orth. because $A^T A$ symm.

$\sigma_i = \sqrt{\lambda_i}$ $\quad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$

$A = U \Sigma V^T \rightsquigarrow U = A V \Sigma^{-1}$

$U^T U = \Sigma^{-1} \underbrace{V^T A^T A V}\Sigma^{-1} = \Sigma^{-1} \Sigma^2 \Sigma^{-1} = I$

Demo: Computing the SVD

R-SVD

In-Class Activity: Eigenvalue Computations

In-class activity: Eigenvalue Computations

Outline

Solving Nonlinear Equations

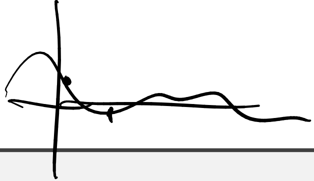
all equation solving can be written like this

What is the goal here?

$$f(\vec{x}) = \vec{0} \qquad f : \mathbb{R}^n \to \mathbb{R}^n$$

$$g(\vec{x}) = \vec{y} \rightsquigarrow f(x) = g(\vec{x}) - \vec{y}$$

Showing Existence



How can we show existence of a root?

1D – Intermediate value thm.

nD – Inverse function thm

$$(f^{-1})' = \frac{1}{f'}$$

f^{-1} helps because $x = f^{-1}(0)$

(guarantees existence of an inverse locally)

Contraction mapping

(terrible)
example
$g(x) := f(x) + x$

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called contractive if there is $0 \leq \gamma < 1$ so that $\|g(x) - g(y)\| \leq \gamma \|x - y\|$.
On a closed set $S \subseteq \mathbb{R}^2$ with $g(S) \subseteq S$, there exists a unique fixed point.

Sensitivity and Multiplicity

What is the sensitivity/conditioning of root finding?

cond(root finding) = cond(evaluating the inverse function)

What are multiple roots?

multiplicity m

$f(x) = (x-1.5)^5$

$f(x)=0$

$f'(x)=0, \ldots, f^{(m-1)}(x)=0$

How do multiple roots interact with conditioning?

multiple roots $\Rightarrow$ terrible conditioning of root finding

Rates of Convergence

What is *linear convergence*? *quadratic convergence*?

$\vec{u}_k$ true solution: $\vec{u}$

$\vec{e}_k = \vec{u}_k - \vec{u}$

An iterative method converges with rate r if

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} >0 \\ <\infty \end{cases}$$

$\|e_{k+1}\| \le C \cdot \|e_k\|^2 \leftarrow$ quadratic

most useful if $0 \le C < 1$ $\quad \|e_{k+1}\| \le C \|e_k\| \leftarrow$ linear

Power iteration: linearly conv.

$$\|x_{k+1} - x\| \le \frac{\lambda_2}{\lambda_1} \cdot \|x_k - x\|$$

About Convergence Rates

Demo: Rates of Convergence

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.