Today:

- constrained opt.
- interpolation

Announcements:

- Exam Let 3

Levenberg-Marquardt

$\|\vec{r}(\vec{x})\|_2$

If Gauss-Newton on its own is poorly, conditioned, can try Levenberg-Marquardt:

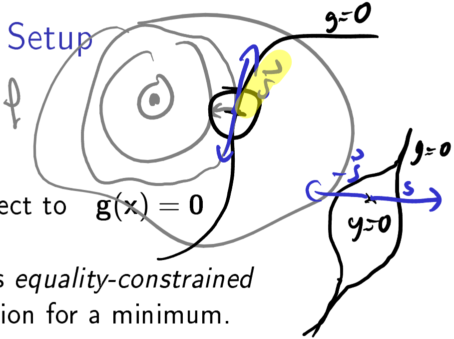
$$J_r^T J_r(x_k)\vec{s}_k = -J^T(x_k)\, r(x_k)$$

$$\Leftrightarrow \quad J(x_k)\,\vec{s}_k \cong -r(x_k)$$

$$J_r^T J_r(x_k)\vec{s}_k + \mu_k I_k s_k = -J^T(x_k)\, r(x_k)$$

$$\Leftrightarrow \quad \begin{bmatrix} J_k(x_k) \\ \sqrt{\mu_k} \end{bmatrix} \vec{s}_k \cong \begin{bmatrix} -r(x_k) \\ 0 \end{bmatrix}$$

Constrained Optimization: Problem Setup



Want $\mathbf{x}^*$ so that

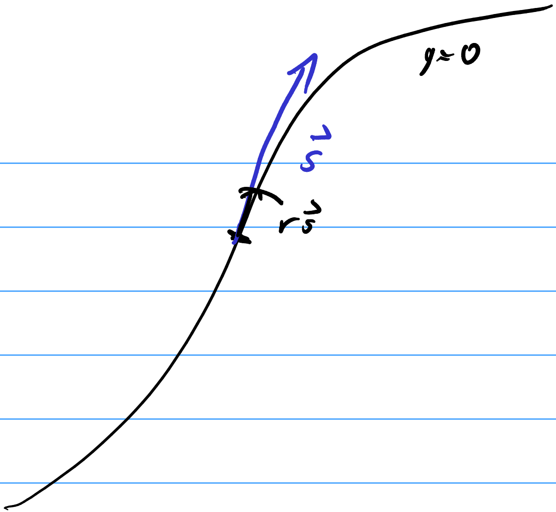
$$f(\mathbf{x}^*) = \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}$$

No inequality constraints just yet. This is *equality-constrained optimization*. Develop a necessary condition for a minimum.

unconstrained: $\nabla f(x^*) = 0$

At x w/ $g(x)=0$, $\vec{s}$ is a feasible direction if I can find an r

so that for $\alpha \in [0, r]$: $g(\vec{x} + \alpha \vec{s}) = 0$



y=0

Constrained Optimization: Necessary Condition

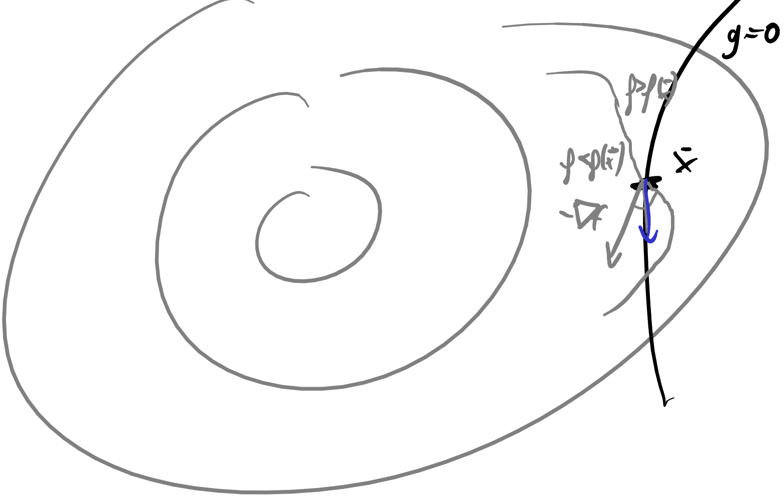
constr min at x^* $\Rightarrow$

$$\nabla f(x^*) \cdot \vec{s} \geq 0$$ "uphill in all feasible directions"

On the interior of $\{g(\vec{x}) = 0\}$, s and $-\vec{s}$ are feasible:

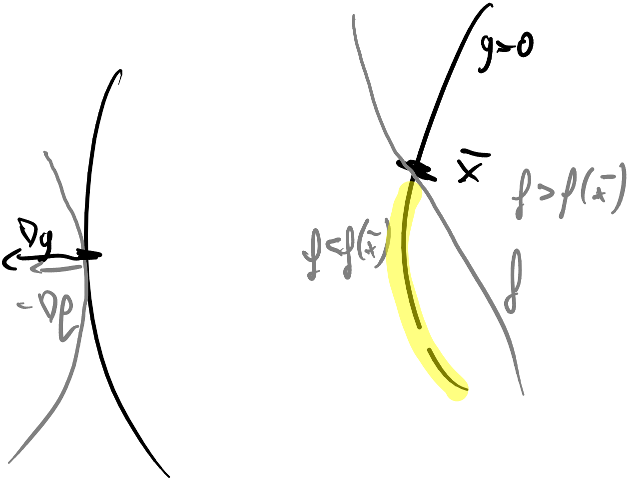
$\nabla f(x^*) \cdot \vec{s} \geq 0$ and $\nabla f(x^*) \cdot (-s) \geq 0$

$$\Rightarrow \nabla f(x^*) = 0$$



$-\nabla f \perp$ level set of g through x^*

$\Leftrightarrow -\nabla f = \lambda \nabla g$



∇g
-∇f
g=0
x̄
f>f(x̄)
f<f(x̄)
f

$g : \mathbb{R}^n \to \mathbb{R}$

$$-\nabla f = \lambda \nabla g$$

$g : \mathbb{R}^n \to \mathbb{R}^2$

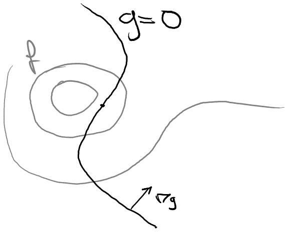
$$J_g : \begin{array}{cc} & n \\ 2 & \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \end{array}$$

$$-\nabla f = \underbrace{\begin{bmatrix} | & | \end{bmatrix}}_{J_g^T} \vec{\lambda}$$

$-\nabla f \in \text{row span}(J_g)$

necessary cond.

Lagrange Multipliers



Seen: Need $-\nabla f(\mathbf{x}) = J_{\mathbf{g}}^T \lambda$ at the (constrained) optimum.

Idea: Turn constrained optimization problem for $\mathbf{x}$ into an *unconstrained* optimization problem for $(\mathbf{x}, \lambda)$. How?

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \lambda^T g(x)$$

Idea: run unconstrained opt.

Lagrange Multipliers: Development

$$\mathcal{L}(\mathbf{x}, \lambda) := f(\mathbf{x}) + \lambda^T \mathbf{g}(\mathbf{x}).$$

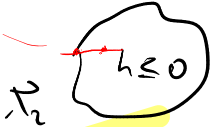
$$0 = \nabla \mathcal{L} = \begin{bmatrix} \nabla_x \mathcal{L} \\ \nabla_\lambda \mathcal{L} \end{bmatrix} = \begin{bmatrix} \nabla f + J_g(x)^T \lambda \\ g(x) \end{bmatrix}$$

For example: use Newton to minimize $\mathcal{L}$

"SQP": sequential quadratic programming

Demo: Sequential Quadratic Programming

Inequality-Constrained Optimization



Want $\mathbf{x}^*$ so that

$$f(\mathbf{x}^*) = \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to} \quad \overset{\lambda_1}{\mathbf{g}(\mathbf{x}) = \mathbf{0}} \quad \text{and} \quad \overset{\lambda_2}{\mathbf{h}(\mathbf{x}) \leqslant \mathbf{0}}$$

This is *inequality-constrained optimization*. Develop a necessary condition for a minimum.

$\boxed{y}$

$\boxed{\tilde{y}} = 0 \Leftrightarrow h(x) \le 0$

$$\bar{g}(x) = \begin{cases} 1 & h(x) > 0 \\ 0 & h(x) \le 0 \end{cases}$$

$$\mathcal{L}(x, \lambda_1, \lambda_2) = f(x) + \lambda_1^T g(x) + \lambda_2^T h(x)$$

ineq. constraints can be active or inactive

i.e. active $\Rightarrow h_i(x) = 0$

$h_i < 0$

If h_i is inactive: must $(\lambda_2)_i = 0$

If h_i is active: Behavior of h_i

$h_i = 0$

could change the minimum

$h_i(x) = 0$

$$(\lambda_2)_i \cdot h_i = 0$$

complementarity condition

$\Rightarrow$ turns h_i "on"/"active" exactly on the bdry.

Inequality-Constrained Optimization (cont'd)

Develop a set of necessary conditions for a minimum.

$$\nabla_x \mathcal{L}(x^*, \lambda_1^*, \lambda_2^*) = 0$$

$$g(x^*) = 0$$

$$\rightarrow \quad h^T(x^*) \cdot \lambda_2 = 0$$

feed to Newton

$$h(x^*) \leq 0$$

$$\rightarrow \quad \lambda_2 \geq 0$$ $\leftarrow$ sign convention on λ_2

Outline