Today

- Interpolation
- Quadrature

Announcements:

- Examlet 4 dates
- Examlet 3 material
- DLMF

Interpolation Error

$n-1$

If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most ~~n~~ that interpolates f at n distinct points $\{x_i\}$ $(i = 1, \ldots, n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$\leq h^4$

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n).$$

error is 0 at the nodes

$$Y(t) = R(t) - \frac{R(x)}{W(x)} W(t)$$

$$R(t) = f(t) - p_{n-1}(t)$$

$$R^{(n)}(t) = f^{(n)}(t) - 0$$

$|x - x_2| \leq h$

$|x - x_1| \leq h$

Interpolation Error: Proof cont'd

$$Y(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

$\underbrace{t = x \to Y(x) = 0}_{1}$ $\underbrace{t = x_i \to Y(x_i) = 0}_{n}$

$\to Y$ has $n+1$ roots

Rolle's theorem

exists

$\to Y^{(n)}(\xi) = 0$

$$Y^{(n)}(t) = \overbrace{f^{(n)}(t)}^{R^{(n)}(t)} - \frac{R(x)}{W(x)} n!$$

$$0 = f^{(n)}(\xi) - \frac{R(x)}{W(x)} n!$$

$$R(x) = \frac{f^{(n)}(\xi)}{n!} w(x)$$

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

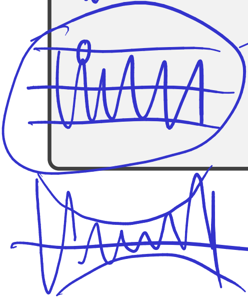
$$\text{error}(x) \approx W(x) = \prod_{i=1}^{n} (x - x_i)$$

If x_i = roots of T_n

$\to$ chebyshev polys

controlled by choosing good nodes

$T_n(x) = \cos(n \cos^{-1}(x))$



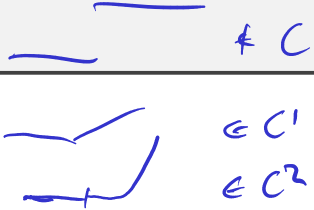
Error Result: Simplified From

Boil the error result down to a simpler form.

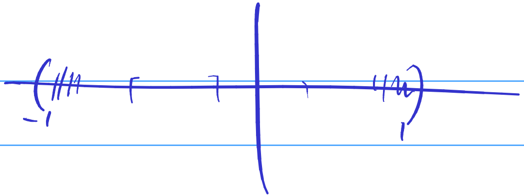
$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} W(x)$$

$$|\text{Interpolation error}| = | \quad \cdots \quad | \leq C(f,n)\, h^n$$

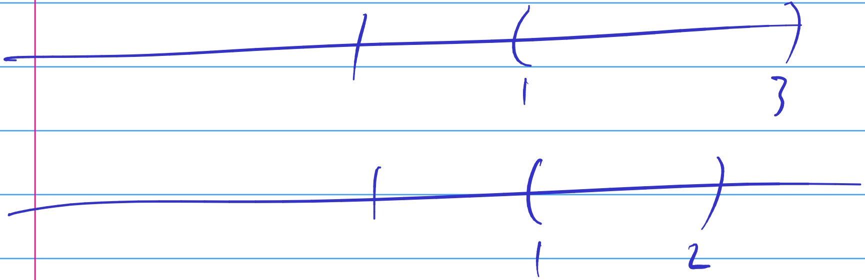
h = length of the interval



Demo: Interpolation Error



Cheb nodes



Going piecewise: Simplest Case

repeat

Construct a piecwise linear interpolant at four points.

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	$f_1 = a_1 x + b_1$		$f_2 = a_2 x + b_2$		$f_3 = a_3 x + b_3$	
	2 unk.		2 unk.		2 unk.	
	$f_1(x_0) = y_0$		$f_2(x_1) = y_1$		$f_3(x_2) = y_2$	
	$f_1(x_1) = y_1$		$f_2(x_2) = y_2$		$f_3(x_3) = y_3$	
	2 eqn.		2 eqn.		2 eqn.	

Why three intervals?

(repeat middle if you need more)

Piecewise Cubic ('Splines')

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	f_1		f_2		f_3	
	$a_1x^3 + b_1x^2 + c_1x + d_1$		$a_2x^3 + b_2x^2 + c_2x + d_2$		$a_3x^3 + b_3x^2 + c_3x + d_3$	

4 unk. 4 unk. 4 unk. $\rightarrow$ 12 unk

$$2 \left[\begin{array}{l} f_1(x_0) = y_0 \\ f_1(x_1) = y_1 \end{array} \right. \quad \rightarrow 6$$

$f_1'(x_1) = f_2'(x_1)$ $\quad$ $f_2'(x_2) = f_3'(x_2)$ $\quad \rightarrow 2$

$f_1''(x_1) = f_2''(x_1)$ $\quad$ $f_2''(x_2) = f_3''(x_2)$ $\quad \rightarrow 2$

Number of cond: $2N_{\text{intervals}} + 2N_{\text{midpts}}$

$$N_{intervals} - 1 = N_{midpoints}$$

\# conds $\quad 2N_{intervals} + 2N_{intervals} - 2$

$$= 4N_{intervals} - 2$$

\# unk: $\quad 4N_{intervals}$

→ need 2 extra extra conditions

→ there are two end pts → one for endpoint

"natural" spline → $f''(x_0) = f''(x_n) = 0$

choice $\rightarrow f'(x_0) = f'(x_3)$

And $f''(x_0) = f''(x_3)$

Piecewise Cubic ('Splines'): Accounting

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	f_1		f_2		f_3	
	$a_1x^3 + b_1x^2 + c_1x + d_1$		$a_2x^3 + b_2x^2 + c_2x + d_2$		$a_3x^3 + b_3x^2 + c_3x + d_3$	

Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems

Nonlinear Equations

Optimization

Interpolation

Numerical Integration and Differentiation
- Numerical Integration
- Quadrature Methods
- Accuracy and Stability
- Composite Quadrature
- Gaussian Quadrature
- Numerical Differentiation
- Richardson Extrapolation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics