Today

- composite quad
- num. diff.
- Richardson extrapolation.
- diff eq. < IVP / BVP

Announcements

- Quiz 24 release ✓
- 4CH assignment 2 ✓
- Examlet 3 debrief ✓
- Examlet 4 content ✓
- Order of accuracy: Gaussian quad.

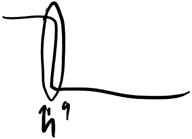
$$\text{Error}(h) = O(h^n)$$

order of accuracy

"If a method is exact for polynomials up to degree $n-1$, then <u>often</u> the order of accuracy is n"

Order of accuracy of Gaussian quad.: $2n$

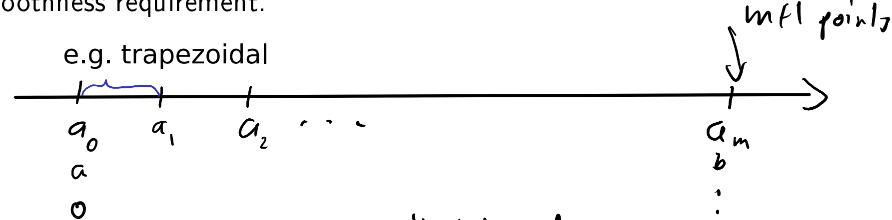
Composite Quadrature



High-order polynomial interpolation requires a high degree of smoothness of the function.

Idea: Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.

e.g. trapezoidal



Quad. error $O(h^q)$

\# intervals $= m$

length of each interval $= 1/m = h$

Error for composite: $\frac{1}{h} O(h^q) = O(h^{q-1})$

↳ composite quad loses an order

Error in Composite Quadrature

What can we say about the error in the case of composite quadrature?

Composite Quadrature: Notes

Observation: Composite quadrature loses an order compared to non-composite.

Idea: If we can estimate errors on each subinterval, we can shrink (e.g. by splitting in half) only those contributing the most to the error. (adaptivity, $\to$ hw)

Taking Derivatives Numerically

$\|\vec{v}\|_\infty = \max |v_i|$

$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$

must be ill-

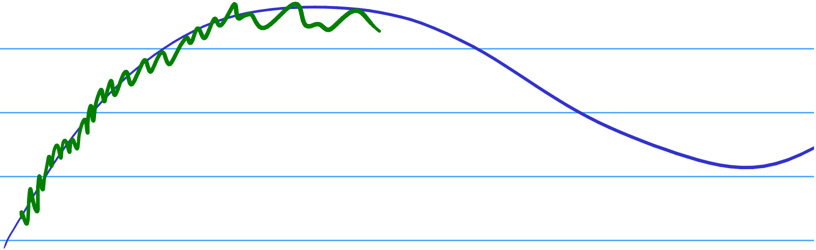
e.g. $\sin(\alpha x)$

Why *shouldn't* you take derivatives numerically?

- unbounded: $\|f\|_\infty \le C \rightarrow \|f'\|_\infty$ can still be unbounded
- has a nullspace $c' = 0$

Demo: Taking Derivatives with Vandermonde Matrices

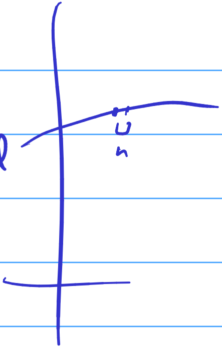
$\|A\|, \|A^{-1}\| \to \infty$



↳ derivative of noise drowns out the derivative of the signal

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

→ cancellation as a numerical method



h

Interpolation : $O(h^n)$

Quadrature : $O(h^{n+1})$

Diff. : $O(h^{n-1})$

Finite Differences

$$f'(x) \approx \cancel{\lim_{h\to 0}} \frac{f(x+h)-f(x)}{h}$$

"finite difference"

$$\boxed{f(x+h) = f(x) + f'(x)h + f''(x)\cdot\frac{h^2}{2} + \cdots}$$

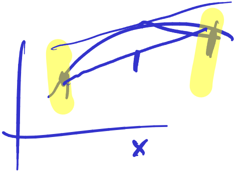
Error: $f''(x)\,\frac{h}{2} + HOT$

$\hookrightarrow$ $\mathcal{O}(h) \to$ first order accurate

More Finite Difference Rules

Similarly:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$



(Centered differences)

Can also take higher order derivatives:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Can find these by trying to match Taylor terms.
Alternative: Use linear algebra with interpolate-then-differentiate to find FD formulas.

Demo: Finite Differences vs Noise

Demo: Floating point vs Finite Differences

$$f(x) \approx \sum_{i=0}^{n-1} \alpha_i \, \varphi_i(x)$$

$$\vec{\alpha} = V^{-1} f(\vec{x}) \rightarrow f(x) = V\vec{\alpha}$$

$$(V')_{ij} = (\varphi_j'(x_i)) \qquad f'(x) \approx \sum_{i=0}^{n-1} \alpha_i \, \varphi_i'(x)$$

$$f'(\vec{x}) = \underbrace{V' V^{-1}}_{\text{differentiation matrix}} f(\vec{x})$$

Richardson Extrapolation

If we have two estimates of something, can we get a third that's more accurate? Suppose we have an approximation $F = \tilde{F}(h) + O(h^p)$ and we know $\tilde{F}(h_1)$ *and* $\tilde{F}(h_2)$.

$$F = \tilde{F}(h) + a\,h^p + O(h^q) \quad \text{often } q = p+1$$

$$F = \alpha\,\tilde{F}(h_1) + \beta\,\tilde{F}(h_2) + O(h^q)$$

$$\alpha \cdot a \cdot h_1^p + \beta \cdot a \cdot h_2^p = 0 \quad / \quad \alpha + \beta = 1$$

$$\alpha = \frac{-h_2^p}{h_1^p - h_2^p} \qquad \alpha = \frac{-\frac{1}{2}}{1 - \frac{1}{2}} = -1$$

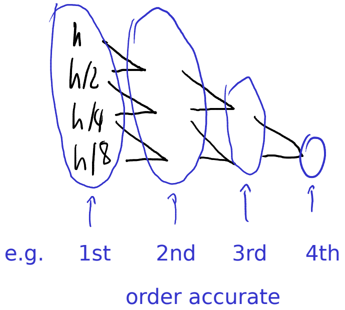
$$\beta = 1 - \alpha$$

$$\frac{1}{2}$$

Richardson Extrapolation: Observations, Romberg Integration

Important observation: Never needed to know a.

Idea: Can repeat this for even higher accuracy.



Carrying out this process for quadrature is called Romberg integration.

Demo: Richardson with Finite Differences

In-Class Activity: Differentiation and Quadrature

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Outline