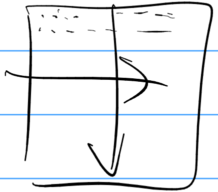
Today

- IVP methods
  accuracy / stability
- BVP

Announcements

- Examlet 4
- Kronecker product



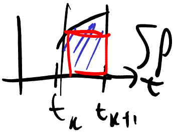
Euler's Method

Discretize the IVP

$$\begin{cases} \mathbf{y}'(t) = \mathbf{f}(\mathbf{y}) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

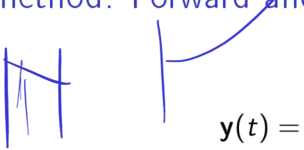
- Discrete times: $t_1, t_2, \ldots,$ with $t_{i+1} = t_i + h$
- Discrete function values: $\mathbf{y}_k \approx \mathbf{y}(t_k)$.

$$y(t_{k+1}) = y_k + \int_{t_k}^{t_{k+1}} f(y(\tau))\, d\tau$$



$\int f(y(\tau))\, d\tau$

Euler's method: Forward and Backward



$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{y}(\tau))\,\mathrm{d}\tau,$$

Stable $y' = \alpha y$, $\mathrm{Re}\,\alpha \le 0$

Unstable $y' = \alpha y$, $\mathrm{Re}\,\alpha > 0$

Use 'left rectangle rule' on integral:

$$y_{k+1} = y_k + h f(y_k)$$

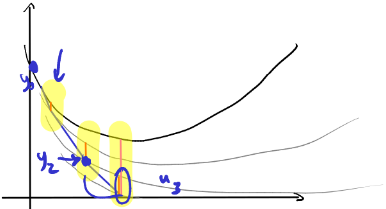
For stable IVP some values of h give unstable

Use 'right rectangle rule' on integral:

"backward Euler" $y_{k+1} = y_k + h f(y_{k+1})$

Demo: Forward Euler stability

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$.

$u_k' = f'(u_k)$

$u_{k-1}(t_{k-1}) = y_{k-1}$

Define the local error at step k as...

$$\boxed{l_k = y_k - u_{k-1}(t_k)}$$

Define the global error at step k as...

$$g_k = y(t_k) - y_k$$

About Local and Global Error

Is global error $= \sum$ local errors?

no.

A time integrator is said to be *accurate of order p* if...

$$\ell_k = O(h^{\underbrace{p+1}})$$

ODE IVP Solvers: Order of Accuracy

A time integrator is said to be *accurate of order p* if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect–why?

because after integrating to time $O(1)$, we've lost an order.

Stability of a Method

$f(y(t))$

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

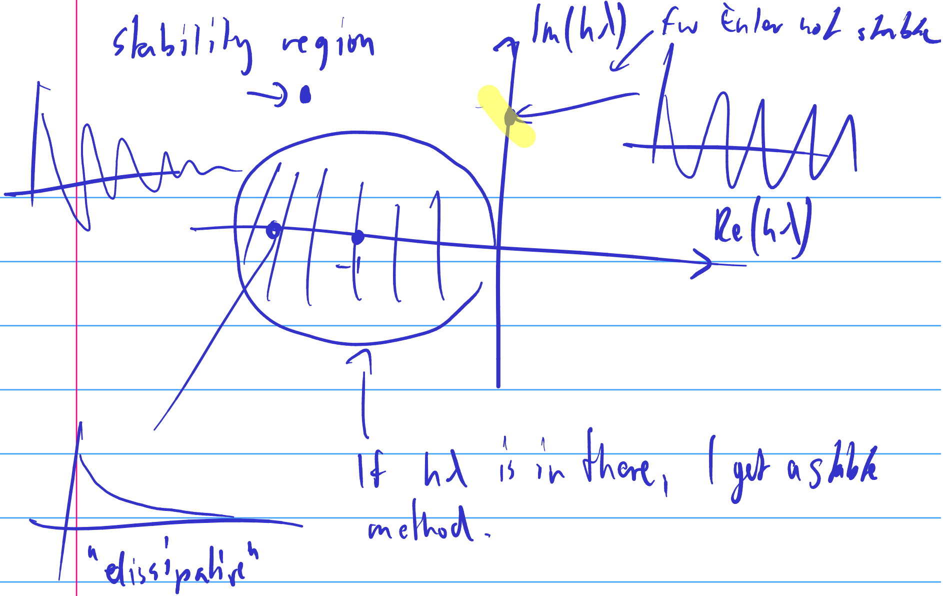
$$y_k = y_{k-1} + h f(y_{k-1})$$

$$= y_{k-1} + h\lambda y_{k-1}$$

$$= (1 + h\lambda)\, y_{k-1}$$

$$(y_n) \text{ stable} \iff |1 + h\lambda| \leq 1$$

$$|1+z| \leq 1 \iff |z - (-1)| \leq 1$$



If $h\lambda$ is in there, I get a stable method.

Stability: Systems

What about stability for systems, i.e.

$$\mathbf{y}'(t) = A\mathbf{y}(t)?$$

$\to$ diagonalize

$\to$ $w_i' = \lambda_i w_i$

stable (if applied to fw Euler)

$\Leftrightarrow |1+h\lambda| \le 1$

Stability: Nonlinear ODEs

What about stability for nonlinear systems, i.e.

$$\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t))?$$

$$e(t) = y(t) - \tilde{y}(t)$$

$$e'(t) = f(y(t)) - f(\tilde{y}(t))$$

$$\approx J_f(y(t)) \underbrace{(\tilde{y}(t) - y(t))}_{e(t)}$$

e (approximately) solves a linear ODE w/ the Jacobian.

IVP is "approximately" stable if eigenvalues λ_i of $\partial f(y(t))$ satisfy $|1+h\lambda| \leq 1$

Stability for Backward Euler

Find out when backward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_k = y_{k-1} + h\lambda y_k$$

$$(1-h\lambda)\, y_k = y_{k-1}$$

$$y_k = \frac{1}{|1-h\lambda|}\, y_{k-1}$$

Demo: Backward Euler stability

lhp: stable ODEs → whole left half plane included

⇒ stable methods
BE

← stab region for bw Euler

whole left half plane $\subseteq$ stab region

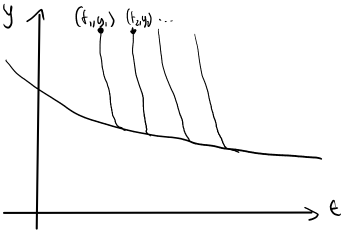
"unconditionally stable"

$$y' = -10{,}000\, y$$

Stiff ODEs: Demo

Demo: Stiffness

'Stiff' ODEs



- Stiff problems have *multiple time scales*.
  (In the example above: Fast decay, slow evolution.)
- In the case of a stable ODE system

$$\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}(t)),$$

stiffness can arise if $J_{\mathbf{f}}$ has eigenvalues of very different magnitude.

Stiffness: Observations

Why not just 'small' or 'large' magnitude?

because ratio matters

What is the problem with applying explicit methods to stiff problems?

need very small Δt

Stiffness vs. Methods

ODE: explicit $y' = f(y)$ implicit $f(y, y') = 0$

methods: explicit $y_{k+1} = g(y_k)$ implicit $0 = g(y_k, y_{k+1})$

Phrase this as a conflict between accuracy and stability.

accuracy: need Δt to be small (implicit + expl.)

stability: implicit lets you take giant Δt and maint. stab.

Can an implicit method take arbitrarily large time steps?

stability says: sure

accuracy: no.

Predictor-Corrector Methods

Idea: Obtain intermediate result, improve it (with same or different method).

For example:

1. *Predict* with forward Euler: $\tilde{y}_{k+1} = y_k + hf(y_k)$
2. *Correct* with the trapezoidal rule: $y_{k+1} = y_k + \frac{h}{2}(f(y_k) + f(\tilde{y}_{k+1}))$.

This is called Heun's method.