Today

- Interpolation
- Quadrature

Announcements:
- Examlet 4 dates
- Examlet 3 material
- OLMF
Interpolation Error

If \( f \) is \( n \) times continuously differentiable on a closed interval \( I \) and \( p_{n-1}(x) \) is a polynomial of degree at most \( n \) that interpolates \( f \) at \( n \) distinct points \( \{x_i\} \) \( (i = 1, \ldots, n) \) in that interval, then for each \( x \) in the interval there exists \( \xi \) in that interval such that

\[
f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2) \cdots (x - x_n).
\]

The error is on the nodes.
Interpolation Error: Proof cont’d

\[ Y(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i) \]

\[ \xi = x \rightarrow Y(x) = 0 \]

\[ \xi = x_i \rightarrow Y(x_i) = 0 \]

\( \xi \) has \( n+1 \) roots

\( \xi \) exists \text{ Rolle’s theorem}

\[ Y^{(n)}(x) = 0 \]

\[ Y^{(n)}(t) = f^{(n)}(t) - \frac{R(x)}{W(x)} n! \]

\( 0 = f^{(n)}(x) - \frac{R(x)}{W(x)} n! \)
\[ R(x') = \frac{f(x')}{\eta} \cdot V(x) \]
Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

\[ \text{error} (x) \propto W(x) = \prod_{i=1}^{n} (x - x_i) \]

\[ T_n(x) = \cos(n \cos^{-1}(x)) \]

If \( x_i \) are roots of \( T_n \)

\( \rightarrow \) chebyshev polys

controlled by choosing good nodes
Boil the error result down to a simpler form.

\[ f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} W(x) \]

\[ \text{Interpolation error} = \frac{1}{C(f^{(n)})} h^n \]

\[ h = \text{length of the interval} \]

**Demo: Interpolation Error**
Going piecewise: Simplest Case

Construct a piecewise linear interpolant at four points.

\[ f_1 = a_1 x + b_1 \quad \text{2 unk.} \]
\[ f_1(x_0) = y_0 \]
\[ f_1(x_1) = y_1 \quad \text{2 eqn.} \]

\[ f_2 = a_2 x + b_2 \quad \text{2 unk.} \]
\[ f_2(x_1) = y_1 \]
\[ f_2(x_2) = y_2 \quad \text{2 eqn.} \]

\[ f_3 = a_3 x + b_3 \quad \text{2 unk.} \]
\[ f_3(x_2) = y_2 \]
\[ f_3(x_3) = y_3 \quad \text{2 eqn.} \]

Why three intervals?

(repeat middle if you need more)
### Piecewise Cubic (‘Splines’)

<table>
<thead>
<tr>
<th>( x_0, y_0 )</th>
<th>( f_1 )</th>
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\[
\begin{align*}
\text{4 unk.} & \\
\begin{cases}
\phi_1'(x_0) = y_0 \\
\phi_1(x_1) = y_1
\end{cases} & \\
\phi_1''(x_1) = \phi_2'(x_1) & \\
\phi_1'''(x_1) = \phi_2''(x_1)
\end{align*}
\]

\[
\begin{align*}
\text{4 unk.} & \\
\phi_2'(x_1) = \phi_3'(x_1) & \\
\phi_2''(x_1) = \phi_3''(x_1)
\end{align*}
\]

Number of cond: \( 2 \times \text{Nintervals} + 2 \times \text{Nmidpts} \)

\( \Rightarrow 2 \times 6 \Rightarrow 12 \) unk.

\( \Rightarrow 6 \)
\[ N_{\text{intervals}} - 1 = N_{\text{midpoints}} \]

\[ 2 N_{\text{intervals}} + 2 N_{\text{intervals}} - 2 = 4 N_{\text{intervals}} - 2 \]

\[ \text{Hence: } 4 N_{\text{intervals}} \]

\[ \rightarrow \text{need 2 extra extra conditions} \]

\[ \rightarrow \text{there are two end pts} \rightarrow \text{one for natural spline: } f''(x_0) - f''(x_s) = 0 \]
choice \rightarrow f'(x_0) = f'(x_3)

And \ f''(x_0) = f''(x_3)
# Piecewise Cubic (‘Splines’): Accounting

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Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems

Nonlinear Equations

Optimization

Interpolation

Numerical Integration and Differentiation
  Numerical Integration
  Quadrature Methods
  Accuracy and Stability
  Composite Quadrature
  Gaussian Quadrature
  Numerical Differentiation
  Richardson Extrapolation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics