Today:

- Nonlinear
- Rates of convergence
- Bisection
- Newton (1D)
- Multiple dim

$\Theta(x) = 0$ smooth
Rates of Convergence

What is linear convergence? quadratic convergence?

\[ x_k \rightarrow x^* \]

\[ e_k = x_k - x^* \]

Not quite:

\[ \|e_k\| \leq C \cdot \|e_{k-1}\| \]

Rate of conv. 1 (linear):

"constant number of accurate digits"

\[ \lim_{k \to \infty} \frac{\|e_k\|}{\|e_{k-1}\|} = C < \infty \]

"linearly convergent"

Rate of conv. 2 (quadratic):

Ideally:

\[ 0 < C < 1 \]
About Convergence Rates

**Demo:** Rates of Convergence
Characterize linear, quadratic convergence in terms of the ‘number of accurate digits’.
Stopping Criteria

Comment on the ‘foolproof-ness’ of these stopping criteria:

1. $|f(x)| < \varepsilon$  (‘residual is small’)
2. $\|x_{k+1} - x_k\| < \varepsilon$
3. $\|x_{k+1} - x_k\| / \|x_k\| < \varepsilon$

Fail when:

1. 

2. 

3. What if $x^* = 0$? 

might stall far away
Bisection Method

\[ \|e_{n+1}\| \leq C \cdot \|e_n\| \]

**Demo:** Bisection Method

What’s the rate of convergence? What’s the constant?

\[ \text{linear} \quad \frac{1}{2} \]
Fixed Point Iteration

\[ x_0 = \text{(starting guess)} \]
\[ x_{k+1} = g(x_k) \]

**Demo:** Fixed point iteration

When does fixed point iteration converge? Assume \( g \) is smooth.
Error in FPI: \( e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \)

\[ e_{k+1} = g'(x_k) e_k \approx q''(x^*) e_k \cdot e_k \]

- Linearly conv. (in nbhd)
- Quadratic conv. (in nbhd)
- \( g''(x^*) < 1 \)
- \( \| g''(x^*) \| = 0 \)

\( g'(x_k) \approx g''(x^*) \cdot (x_k - x^*) \)
Newton’s Method

Derive Newton’s method.
Convergence and Properties of Newton

What’s the rate of convergence of Newton’s method?

Drawbacks of Newton?

Demo: Newton’s method
Demo: Convergence of Newton’s Method