CS 450: Numerical Analysis
Linear Systems

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1 These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath (slides).
Vector Norms

- Properties of vector norms

- A norm is uniquely defined by its unit sphere:

- $p$-norms
Inner-Product Spaces

- **Properties of inner-product spaces:** Inner products $\langle x, y \rangle$ must satisfy

  \[
  \langle x, x \rangle \geq 0
  \]

  \[
  \langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0
  \]

  \[
  \langle x, y \rangle = \langle y, x \rangle
  \]

  \[
  \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle
  \]

  \[
  \langle \alpha x, y \rangle = \alpha \langle x, y \rangle
  \]

- **Inner-product-based vector norms**
Matrix Norms

Properties of matrix norms:

\[ \|A\| \geq 0 \]
\[ \|A\| = 0 \iff A = 0 \]
\[ \|\alpha A\| = |\alpha| \cdot \|A\| \]
\[ \|A + B\| \leq \|A\| + \|B\| \quad \text{(triangle inequality)} \]

Frobenius norm:

Operator/induced/subordinate matrix norms:
Induced Matrix Norms

- Interpreting induced matrix norms:

- General induced matrix norms:
Matrix Condition Number

Definition: \( \kappa(A) = \|A\| \cdot \|A^{-1}\| \) is the ratio between the shortest/longest distances from the unit-ball center to any point on the surface.

Intuitive derivation:

\[
\kappa(A) = \max_{\text{inputs}} \max_{\text{perturbations in input}} \left| \frac{\text{relative perturbation in output}}{\text{relative perturbation in input}} \right|
\]

since a matrix is a linear operator, we can decouple its action on the input \( x \) and the perturbation \( \delta x \) since \( A(x + \delta x) = Ax + A\delta x \), so

\[
\kappa(A) = \frac{\max_{\text{perturbations in input}} \|A\|}{\max_{\text{inputs}} \text{relative perturbation growth}} \frac{\max_{\text{relative input reduction}} 1/\|A^{-1}\|}{1/\|A^{-1}\|}
\]
Matrix Conditioning

- The matrix condition number $\kappa(A)$ is the ratio between the max and min distance from the surface to the center of the unit ball transformed by $\kappa(A)$:

- The matrix condition number bounds the worst-case amplification of error in a matrix-vector product:
Norms and Conditioning of Orthogonal Matrices

- Orthogonal matrices:

- Norm and condition number of orthogonal matrices:
Singular Value Decomposition

- The singular value decomposition (SVD):
Norms and Conditioning via SVD

- Norm and condition number in terms of singular values:
Visualization of Matrix Conditioning

\[ \{ x : x \in \mathbb{R}^2, \| x \|_2 = 1 \} \xrightarrow{A} \{ Ax : x \in \mathbb{R}^2, \| x \|_2 = 1 \} \]

\[ \kappa(A) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \]

\[ A = U \begin{bmatrix} \sigma_{\text{max}} \\ \sigma_{\text{min}} \end{bmatrix} V^T \]

\[ \| A \|_2 = \sigma_{\text{max}} \]

\[ 1/\| A^{-1} \|_2 = \sigma_{\text{min}} \]
Conditioning of Linear Systems

- Let's now return to formally deriving the conditioning of solving $Ax = b$: 
Consider perturbations to the input coefficients $\hat{A} = A + \delta A$: 
Solving Basic Linear Systems

- Solve $Dx = b$ if $D$ is diagonal

- Solve $Qx = b$ if $Q$ is orthogonal

- Given SVD $A = UΣV^T$, solve $Ax = b$
Solving Triangular Systems

- $Lx = b$ if $L$ is lower-triangular is solved by forward substitution:

  
  \[
  \begin{align*}
  l_{11}x_1 &= b_1 \\
  l_{21}x_1 + l_{22}x_2 &= b_2 \quad \Rightarrow \quad x_2 = \\
  l_{31}x_1 + l_{32}x_2 + l_{33}x_3 &= b_3 \quad x_3 = \\
  \vdots & \quad \vdots
  \end{align*}
  \]

- Algorithm can also be formulated recursively by blocks:
Solving Triangular Systems

- Existence of solution to $Lx = b$:

- Uniqueness of solution:

- Computational complexity of forward/backward substitution:
Properties of Triangular Matrices

- $Z = XY$ is lower triangular is $X$ and $Y$ are both lower triangular:

- $L^{-1}$ is lower triangular if it exists:
LU Factorization

- An **LU factorization** consists of a unit-diagonal lower-triangular factor $L$ and upper-triangular factor $U$ such that $A = LU$:

- Given an LU factorization of $A$, we can solve the linear system $Ax = b$: 
Gaussian Elimination Algorithm

- Algorithm for factorization is derived from equations given by \( A = LU \):

- The computational complexity of LU is \( O(n^3) \):
Existence of LU Factorization

- The LU factorization may not exist: Consider matrix \[
\begin{pmatrix}
3 & 2 \\
6 & 4 \\
0 & 3
\end{pmatrix}.
\]

- Permutation of rows enables us to transform the matrix so the LU factorization does exist:
Gaussian Elimination with Partial Pivoting

- **Partial pivoting** permutes rows to make divisor $u_{ii}$ is maximal at each step:

A row permutation corresponds to an application of a row permutation matrix $P_{jk} = I - (e_j - e_k)(e_j - e_k)^T$:
Partial Pivoting Example

Let's consider again the matrix $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \\ 0 & 3 \end{bmatrix}$. 
Complete Pivoting

- *Complete pivoting* permutes rows and columns to make divisor $u_{ii}$ maximal at each step:

- Complete pivoting is noticeably more expensive than partial pivoting:
Round-off Error in LU

- Lets consider factorization of \[
\begin{bmatrix}
\epsilon & 1 \\
1 & 1
\end{bmatrix}
\]
where \( \epsilon < \epsilon_{\text{mach}} \):

- Permuting the rows of \( A \) in partial pivoting gives \( PA = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \)
Error Analysis of LU

The main source of round-off error in LU is in the computation of the Schur complement:

\[ \hat{L} \hat{U} = A + \delta A \]

When computed in floating point, absolute backward error \( \delta A \) in LU is

\[ |\delta a_{ij}| \leq \epsilon_{\text{mach}} (|\hat{L}| \cdot |\hat{U}|)_{ij} \]
Helpful Matrix Properties

- Matrix is *diagonally dominant*, so \( \sum_{i \neq j} |a_{ij}| \leq |a_{ii}|. \)

- Matrix is *symmetric positive definite (SPD)*, so \( \forall x \neq 0, x^T Ax > 0. \)

- Matrix is symmetric but indefinite:

- Matrix is *banded*, \( a_{ij} = 0 \) if \( |i - j| > b. \)
Solving Many Linear Systems

Suppose we have computed \( A = LU \) and want to solve \( AX = B \) where \( B \) is \( n \times k \) with \( k < n \):

Suppose we have computed \( A = LU \) and now want to solve a perturbed system \( (A - uv^T)x = b \):

Can use the Sherman-Morrison-Woodbury formula

\[
(A - uv^T)^{-1} = A^{-1} + \frac{A^{-1}uv^TA^{-1}}{1 - v^TA^{-1}u}
\]