CS 450: Numerical Analysis

Interpolation

University of Illinois at Urbana-Champaign

---

1 These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath (slides).
Interpolation

- Given \((t_1, y_1), \ldots, (t_m, y_m)\) with nodes \(t_1 < \cdots < t_m\) an interpolant \(f\) satisfies:

- Interpolant is usually constructed as linear combinations of \textit{basis functions} \(\{\phi_j\}_{j=1}^n = \phi_1, \ldots, \phi_n\) so

\[
f(t) = \sum_j x_j \phi_j(t).
\]
Polynomial Interpolation

- The choice of \textit{monomials} as basis functions, \( \phi_j(t) = t^{j-1} \) yields a degree \( n - 1 \) polynomial interpolant:

- Polynomial interpolants are easy to evaluate and do calculus on:
Conditioning of Interpolation

- Conditioning of interpolation matrix $A$ depends on basis functions and coordinates $t_1, \ldots, t_m$:

- The Vandermonde matrix tends to be ill-conditioned:
Lagrange Basis

- $n$-points fully define the unique $(n - 1)$-degree polynomial interpolant in the Lagrange basis:

- Lagrange polynomials yield an ideal Vandermonde system, but the basis functions are hard to evaluate and do calculus on:
The Newton basis functions $\phi_j(t) = \prod_{k=1}^{j-1}(t - t_k)$ with $\phi_1(t) = 1$ seek the best of monomial and Lagrange bases:

The Newton basis yields a triangular Vandermonde system:
Orthogonal Polynomials

Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of \textit{orthonormal functions}:
Legendre Polynomials

The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:

The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with $w(t) = \begin{cases} 1 : -1 \leq t \leq 1 \\ 0 : \text{otherwise} \end{cases}$ and normalized so $\hat{\phi}_i(1) = 1$. 

Demo: Orthogonal Polynomials
Chebyshev Basis

Chebyshev polynomials $\phi_j(t) = \cos((j - 1) \arccos(t))$ and Chebyshev nodes $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ provide a way to pick nodes $t_1, \ldots, t_n$ along with a basis, to yield perfect conditioning:
Chebyshev Nodes Intuition

▶ Note *equi-oscillation* property, successive extrema of $T_k = \phi_k$ have the same magnitude but opposite sign.

▶ Set of $k$ Chebyshev nodes of are given by zeros of $T_k$ and are abscissas of points uniformly spaced on the unit circle.
Error in Interpolation

We show by induction that given degree $n$ polynomial interpolant $\tilde{f}$ of $f$ the error $E(t) = f(t) - \tilde{f}(t)$ has $n$ zeros $t_1, \ldots, t_n$ and there exist $y_1, \ldots, y_n$ so

$$E(t) = \int_{t_1}^{t} \int_{y_1}^{w_0} \cdots \int_{y_n}^{w_{n-1}} f^{(n+1)}(w_n) dw_n \cdots dw_0 \quad (1)$$
Interpolation Error Bounds

Consequently, polynomial interpolation satisfies the following error bound:

Letting $h = t_n - t_1$ (often also achieve same for $h$ as the node-spacing $t_{i+1} - t_i$), we obtain
Piecewise Polynomial Interpolation

- The $k$th piece of the interpolant is typically chosen as polynomial on $[t_i, t_{i+1}]$.

**Hermite** interpolation ensures consecutive interpolant pieces have same derivative at each *knot* $t_i$:
Spline Interpolation

A spline is a \((k - 1)\)-time differentiable piecewise polynomial of degree \(k\):

The resulting interpolant coefficients are again determined by an appropriate generalized Vandermonde system:
B-Splines

**B-splines** provide an effective way of constructing splines from a basis:

- The basis functions can be defined recursively with respect to degree:

  - \( \phi_1 \) is a linear hat function that increases from 0 to 1 on \([t_i, t_{i+1}]\) and decreases from 1 to 0 on \([t_{i+1}, t_{i+2}]\).

  - \( \phi_k \) is positive on \([t_i, t_{i+k+1}]\) and zero elsewhere.

  - The B-spline basis spans all possible splines of degree \( k \) with nodes \( \{t_i\}_{i=1}^n \).

  - The B-spline basis coefficients are determined by a Vandermonde system that is lower-triangular and banded (has \( k \) subdiagonals), and need not contain differentiability constraints, since \( f(t) \) is a sum of \( \phi_k \)'s.