CS 450: Numerical Analysis

Initial Value Problems for Ordinary Differential Equations

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Ordinary Differential Equations

- An ordinary differential equation (ODE) usually describes time-varying system by a function $y(t)$ that satisfies a set of equations in its derivatives.

- An ODE of any order $k$ can be transformed into a first-order ODE,
Example: Newton’s Second Law

- $F = ma$ corresponds to a second order ODE,

- We can transform it into a first order ODE in two variables:
Initial Value Problems

- Generally, a first order ODE specifies only the derivative, so the solutions are non-unique. An *initial condition* addresses this:

Given an initial condition, an ODE must satisfy an integral equation for any given point \( t \):
Existence and Uniqueness of Solutions

- For an ODE to have a unique solution, it must be defined on a closed domain $D$ and be *Lipschitz continuous*:

- The solutions of an ODE can be stable, unstable, or asymptotically stable:
Stability of 1D ODEs

- The solution to the scalar ODE $y' = \lambda y$ is $y(t) = y_0 e^{\lambda t}$, with stability dependent on $\lambda$:

- A constant-coefficient linear ODE has the form $y' = Ay$, with stability dependent on the real parts of the eigenvalues of $A$: 
Numerical Solutions to ODEs

Methods for numerical ODEs seek to approximate \( y(t) \) at \( \{t_k\}_{k=1}^m \).

Euler’s method provides the simplest method (attempt) for obtaining a numerical solution:
Error in Numerical Methods for ODEs

- Truncation error is typically the main quantity of interest, which can be defined *globally* or *locally*:

- The *order of accuracy* of a given method is one less than than the order of the leading order term in the local error $l_k$:
Accuracy and Taylor Series Methods

- By taking a degree-$r$ Taylor expansion of the ODE in $t$, at each consecutive $(t_k, y_k)$, we achieve $r$th order accuracy.

- Taylor series methods require high-order derivatives at each step:
Growth Factors and Stability Regions

- Stability of an ODE method discerns whether local errors are amplified, deamplified, or stay constant:

- Basic stability properties follow from analysis of linear scalar ODE, which serves as a local approximation to more complex ODEs.
Stability Region for Forward Euler

- The stability region of a general ODE constrains the eigenvalues of $hJ_f$
Implicit methods for ODEs form a sequence of solutions that satisfy conditions on a local approximation to the solution:

The stability region of the backward Euler method is the left half of the complex plane:
Trapezoid Method

- A second-order accurate implicit method is the trapezoid method

- Generally, methods can be derived from quadrature rules:
Multi-Stage Methods

- **Multi-stage methods** construct $y_{k+1}$ by approximating $y$ between $t_k$ and $t_{k+1}$:

$$y_{k+1} = y_k + (h/6)(v_1 + 2v_2 + 2v_3 + v_4)$$

$$v_1 = f(t_k, y_k), \quad v_2 = f(t_k + h/2, y_k + (h/2)v_1),$$

$$v_3 = f(t_k + h/2, y_k + (h/2)v_2), \quad v_4 = f(t_k + h, y_k + hv_3).$$

- The 4th order Runge-Kutta scheme is particularly popular:

  *This scheme uses Simpson’s rule,*

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Runge-Kutta Methods

- Runge-Kutta methods evaluate $f$ at $t_k + c_i h$ for $c_0, \ldots, c_r \in [0, 1]$,

$$u_k(t_{k+1}) = y_k + \int_{t_k}^{t_k + h} f(s, y(s)) \, ds \approx y_k + h \sum_{i=0}^{r-1} w_i f(t_k + c_i h, \hat{y}_{ki}),$$

- A general family of Runge Kutta methods can be defined by

$$\hat{y}_{ki} = y_k + h \sum_j a_{ij} f(t_k + c_i h, \hat{y}_{kj}).$$
Runge-Kutta methods are **self-starting**, but are harder to use to obtain error estimates.

- **Self-starting means that we only need** $y_k$ **to form** $y_{k+1}$.

- **Embedded Runge-Kutta schemes provides 4th + 5th order results, yielding an error estimate.**

**Extrapolation methods** achieve high accuracy by successively reducing step-size.

*Use single-step method with step sizes* $h, h/2, h/4, ...$ *to approximate solution at* $t_k + h$. 

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**Properties of Runge-Kutta and Extrapolation Methods**

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Multistep Methods

- **Multistep methods** employ $\{y_k\}_{i=0}^k$ to compute $y_{k+1}$:

- Multistep methods are not self-starting, but have practical advantages: