CS 450: Numerical Analysis

Boundary Value Problems for Ordinary Differential Equations

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1 These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath (slides).
Boundary Conditions

Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary.

Consider a first order ODE \( y'(t) = f(t, y) \) with linear boundary conditions on domain \( t \in [a, b] \):

\[
B_a y(a) + B_b y(b) = c
\]
Existence of Solutions for Linear ODE BVPs

The solutions of linear ODE BVP $y'(t) = A(t)y(t) + b(t)$ are linear combinations of solutions to linear homogeneous ODE IVPs $y'(t) = A(t)y(t)$:

Solution $u(t)$ (and $y(t)$) exists if $Q = B_a Y(a) + B_b Y(b)$ is invertible:
For any given $b(t)$ and $c$, the solution to the BVP can be written in the form:

$$y(t) = \Phi(t)c + \int_a^b G(t, s)b(s)ds$$

$\Phi(t) = Y(t)Q^{-1}$ is the fundamental matrix and the Green's function is

$$G(t, s) = Y(t)Q^{-1}I(s)Y^{-1}(s), \quad I(s) = \begin{cases} B_aY(a) & : s < t \\ -B_bY(b) & : s \geq t \end{cases}$$
Conditioning of Linear ODE BVPs

- For any given $b(t)$ and $c$, the solution to the BVP can be written in the form:

$$y(t) = \Phi(t)c + \int_a^b G(t,s)b(s)ds$$

- The absolute condition number of the BVP is $\kappa = \max\{\|\Phi\|_\infty, \|G\|_\infty\}$:
For linear ODEs, we construct solutions from IVP solutions in $Y(t)$, which suggests the \textit{shooting method} for solving BVPs by reduction to IVPs:

\textit{Multiple shooting} employs the shooting method over subdomains:
Finite Difference Methods

- Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

- Convergence to solution is obtained with decreasing step size $h$ so long as the method is consistent and stable:
Finite Difference Methods

Let's derive the finite difference method for the ODE BVP defined by

\[ u'' + 7(1 + t^2)u = 0 \]

with boundary conditions \( u(-1) = 3 \) and \( u(1) = -3 \), using a centered difference approximation for \( u'' \) on \( t_1, \ldots, t_n, t_{i+1} - t_i = h \).
Collocation Methods

- **Collocation methods** approximate $y$ by representing it in a basis

$$y(t) \approx v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t).$$

- Choices of basis functions give different families of methods:
Solving BVPs by Optimization

- To improve robustness, define and minimize a residual error over the whole domain rather than at collocation points.

- The first-order optimality conditions of the optimization problem are a system of linear equations $Ax = b$: 
Weighted Residual

- *Weighted residual methods* work by ensuring the residual is orthogonal with respect to a given set of weight functions:

- The Galerkin method is a weighted residual method where $w_i = \phi_i$. 
Second-Order BVPs: Poisson Equation

In practice, BVPs are at least second order and it's advantageous to work in the natural set of variables.

Consider the *Poisson equation* \( u'' = f(t) \) with boundary conditions \( u(a) = u(b) = 0 \) and define a localized basis of hat functions:

Defining residual equation by analogy to the first order case, we obtain,
Weak Form and the Finite Element Method

- The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in *weak form*: 

  \[ \text{weak form} \]
A typical second-order scalar ODE BVP eigenvalue problem is

$$u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0.$$ 

These can be solved, e.g. for $f(t, u, u') = g(t)u$ by finite differences:
Generalized matrix eigenvalue problems arise from more sophisticated ODEs,

\[ u'' = \lambda (g(t)u + h(t)u'), \quad \text{with boundary conditions} \quad u(a) = 0, u(b) = 0. \]