# CS 450: Numerical Anlaysis[1]

## Boundary Value Problems for Ordinary Differential Equations

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[1] *These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book "Scientific Computing: An Introductory Survey" by Michael T. Heath (slides).*

# Boundary Conditions

- Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary.

- Consider a first order ODE $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y})$ with *linear boundary conditions* on domain $t \in [a, b]$:

$$\boldsymbol{B}_a \boldsymbol{y}(a) + \boldsymbol{B}_b \boldsymbol{y}(b) = \boldsymbol{c}$$

# Existence of Solutions for Linear ODE BVPs

- The solutions of linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ are linear combinations of solutions to linear homogeneous ODE IVPs $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t)$:

- Solution $\boldsymbol{u}(t)$ (and $\boldsymbol{y}(t)$) exists if $\boldsymbol{Q} = \boldsymbol{B}_a\boldsymbol{Y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b)$ is invertible:

## Green's Function Form of Solution for Linear ODE BVPs

- For any given $\boldsymbol{b}(t)$ and $\boldsymbol{c}$, the solution to the BVP can be written in the form:

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)ds$$

$\boldsymbol{\Phi}(t) = \boldsymbol{Y}(t)\boldsymbol{Q}^{-1}$ is the *fundamental matrix* and the *Green's function* is

$$\boldsymbol{G}(t,s) = \boldsymbol{Y}(t)\boldsymbol{Q}^{-1}\boldsymbol{I}(s)\boldsymbol{Y}^{-1}(s), \quad \boldsymbol{I}(s) = \begin{cases} \boldsymbol{B}_a\boldsymbol{Y}(a) & : s < t \\ -\boldsymbol{B}_b\boldsymbol{Y}(b) & : s \geq t \end{cases}$$

# Conditioning of Linear ODE BVPs

- For any given $\boldsymbol{b}(t)$ and $\boldsymbol{c}$, the solution to the BVP can be written in the form:

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)ds$$

- The absolute condition number of the BVP is $\kappa = \max\{||\boldsymbol{\Phi}||_\infty, ||\boldsymbol{G}||_\infty\}$:

# Shooting Method for ODE BVPs

**Demo:** *Shooting method*

- For linear ODEs, we construct solutions from IVP solutions in $\boldsymbol{Y}(t)$, which suggests the *shooting method* for solving BVPs by reduction to IVPs:

- *Multiple shooting* employs the shooting method over subdomains:

# Finite Difference Methods

- Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:
- Convergence to solution is obtained with decreasing step size $h$ so long as the method is consistent and stable:

# Finite Difference Methods

- Lets derive the finite difference method for the ODE BVP defined by

$$u'' + 7(1+t^2)u = 0$$

with boundary conditions $u(-1) = 3$ and $u(1) = -3$, using a centered difference approximation for $u''$ on $t_1, \ldots, t_n$, $t_{i+1} - t_i = h$.

# Collocation Methods

*Demo: Sparse matrices*

- *Collocation methods* approximate $\boldsymbol{y}$ by representing it in a basis

$$\boldsymbol{y}(t) \approx \boldsymbol{v}(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \boldsymbol{\phi}_i(t).$$

- Choices of basis functions give different families of methods:

# Solving BVPs by Optimization

- To improve robustness, define and minimize a residual error over the whole domain rather than at collocation points.

- The first-order optimality conditions of the optimization problem are a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$:

# Weighted Residual

- *Weighted residual methods* work by ensuring the residual is orthogonal with respect to a given set of weight functions:

- The *Galerkin method* is a weighted residual method where $\boldsymbol{w}_i = \phi_i$.

# Second-Order BVPs: Poisson Equation

In practice, BVPs are at least second order and its advantageous to work in the natural set of variables.

- Consider the *Poisson equation* $u''(t) = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:
- Defining residual equation by analogy to the first order case, we obtain,

## Weak Form and the Finite Element Method

- The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in *weak form*:

# Eigenvalue Problems with ODEs

- A typical second-order scalar *ODE BVP eigenvalue problem* is to find eigenvalue $\lambda$ and eigenfunction $u$ to satisfy

  $$u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } \; u(a) = 0, u(b) = 0.$$

  These can be solved, e.g. for $f(t, u, u') = g(t)u$ by finite differences:

# Using Generalized Matrix Eigenvalue Problems

- Generalized matrix eigenvalue problems arise from more sophisticated ODEs,

$$u'' = \lambda(g(t)u + h(t)u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0.$$