

CS 450: Numerical Analysis¹

Fast Fourier Transform

University of Illinois at Urbana-Champaign

¹ *These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath ([slides](#)).*

Sparse Linear Systems and Time-independent PDEs

- ▶ The Poisson equation serves as a model problem for numerical methods:
 - ▶ *the 2D Poisson problem and resulting Kronecker product linear system are a common benchmark,*
 - ▶ *this system has the form $\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T}$ where $\mathbf{T}$ is tridiagonal.*
- ▶ Dense, sparse direct, iterative, FFT, and Multigrid methods provide increasingly good complexity for the problem:
 - ▶ *dense linear system solve costs $O(n^3)$ naively,*
 - ▶ *nested dissection with Cholesky has $O(n^{3/2})$ complexity and $O(n \log n)$ memory*
 - ▶ *Conjugate-Gradient gives $O(n^{3/2})$ complexity with $O(n)$ memory*
 - ▶ *FFT achieves $O(n \log n)$ cost and multigrid achieves $O(n)$.*

Multigrid

- ▶ Multigrid employs a hierarchy of grids to accelerate iterative methods:
 - ▶ *the residual equation $A\hat{x} = r$ on each fine grid, is approximately solved on the next coarser grid,*
 - ▶ *the equation is **restricted** by projection matrix P , so that $PAP^T P\hat{x} = Pr$*
 - ▶ *the interpolation operator (often given by P^T) is used to obtain an approximate $\hat{x}$ based on the coarse grid approximate solution,*
 - ▶ *at each level we perform some smoothing operations (e.g. Jacobi or Conjugate Gradient) before restriction and after interpolation,*
 - ▶ *at the coarsest level we typically solve directly.*
- ▶ The multigrid method works by resolving high-frequency error components on finer-grids and low-frequency error components on coarser grids:
 - ▶ *smoothers are usually effective at reducing local error, but slow at resolving global (low-frequency) components of the error,*
 - ▶ *on coarser grids, the low frequency error may be resolved more quickly.*

Multigrid

- Consider the Galerkin approximation with linear finite elements to the Poisson equation $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$:

$$\phi_i^{(h)}(t) = \begin{cases} (t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\ (t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\ 0 & : \text{otherwise} \end{cases}$$

where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$. *The weak form with grid spacing of h is*

$$\int_a^b f(t) \phi_i^{(h)}(t) dt = - \sum_{j=1}^n x_j \int_a^b \phi_j^{(h)'}(t) \phi_i^{(h)'}(t) dt.$$

in multigrid, we define a coarse grid basis of $(n-1)/2$ functions, which are hat functions of twice the width,

$$\phi_i^{(2h)}(t) = \frac{1}{2} \phi_{2i-2}^{(h)}(t) + \phi_{2i-1}^{(h)}(t) + \frac{1}{2} \phi_{2i}^{(h)}(t) = \begin{cases} (t - t_{i-2})/2h & : t \in [t_{i-2}, t_i] \\ (t_{i+2} - t)/2h & : t \in [t_i, t_{i+2}] \\ 0 & : \text{otherwise} \end{cases}$$

Coarse Grid Matrix

- Multigrid restricts the residual equation on the fine grid $\mathbf{A}^{(h)}\mathbf{x} = \mathbf{r}^{(h)}$ to the coarse grid: Let $\phi^{(2h)} = \begin{bmatrix} \phi_1^{(2h)} & \cdots & \phi_{(n-1)/2}^{(2h)} \end{bmatrix}$ and $\phi^{(h)} = \begin{bmatrix} \phi_1^{(h)} & \cdots & \phi_n^{(h)} \end{bmatrix}$ and define *restriction matrix* $\mathbf{P}$ so that $\phi^{(2h)} = \mathbf{P}\phi^{(h)}$, i.e.,

$$\mathbf{P} = \frac{1}{2} \begin{bmatrix} 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{(1)} \\ \mathbf{p}^{(2)} \\ \vdots \end{bmatrix}.$$

The coarse grid stiffness matrix is given by

$$\begin{aligned} a_{ij}^{(2h)} &= - \int_a^b \phi_j^{(2h)'}(t) \phi_i^{(2h)'}(t) dt \\ &= - \mathbf{p}^{(i)} \underbrace{\left(\int_a^b \phi^{(h)'}(t) \phi^{(h)'}{}^T(t) dt \right)}_{-\mathbf{A}^{(h)}} \mathbf{p}^{(j)T}, \end{aligned}$$

$$\mathbf{A}^{(2h)} = \mathbf{P} \mathbf{A}^{(h)} \mathbf{P}^T.$$

Restricting the Residual Equation

- ▶ Given the fine-grid residual $\mathbf{r}^{(h)}$, we seek to use the coarse grid to approximate $\mathbf{x}^{(h)}$ so that $\mathbf{A}\mathbf{x}^{(h)} \approx \mathbf{r}^{(h)}$
 - ▶ *Given a function in the coarse grid basis, $u^{(2h)} = \mathbf{x}^{(2h)T} \boldsymbol{\phi}^{(2h)}$, we can express it in the fine-grid basis via*

$$u^{(2h)} = \mathbf{x}^{(2h)T} \underbrace{\mathbf{P}\boldsymbol{\phi}^{(h)}}_{\boldsymbol{\phi}^{(2h)}} = \underbrace{\mathbf{x}^{(2h)T} \mathbf{P}}_{\mathbf{x}^{(h)T}} \boldsymbol{\phi}^{(h)}.$$

- ▶ *Consequently, the solution to the restricted residual equation $\mathbf{A}^{(2h)}\mathbf{x}^{(2h)} = \mathbf{r}^{(2h)}$ will lead to an approximate residual equation solution on the fine grid with $\mathbf{x}^{(h)} = \mathbf{P}^T \mathbf{x}^{(2h)}$.*
- ▶ *Noting this, we derive the form of the coarse grid residual,*

$$\begin{aligned}\mathbf{r}^{(2h)} &= \mathbf{A}^{(2h)}\mathbf{x}^{(2h)} \\ &= \mathbf{P}\mathbf{A}^{(h)}\mathbf{P}^T \mathbf{x}^{(2h)} = \mathbf{P}\mathbf{A}^{(h)}\mathbf{x}^{(h)} \\ &= \mathbf{P}\mathbf{r}^{(h)}.\end{aligned}$$

Discrete Fourier Transform

- ▶ The solutions to hyperbolic PDEs like Poisson are wave-like and take on simple representations in the frequency basis, both for continuous and discretized equations. We define the *discrete Fourier transform* using

$$\omega_{(n)} = \cos(2\pi/n) - i \sin(2\pi/n) = e^{-2\pi i/n}.$$

The DFT matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$ is given by $f_{ij} = \omega_{(n)}^{ij}$,

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_{(4)}^1 & \omega_{(4)}^2 & \omega_{(4)}^3 \\ 1 & \omega_{(4)}^2 & \omega_{(4)}^4 & \omega_{(4)}^6 \\ 1 & \omega_{(4)}^3 & \omega_{(4)}^6 & \omega_{(4)}^9 \end{bmatrix}$$

- ▶ it is complex and symmetric (not Hermitian),
- ▶ it is unitary modulo scaling $\mathbf{F}^* = n\mathbf{F}^{-1}$.

The discrete Fourier transform of vector $\mathbf{v}$ is $\mathbf{F}\mathbf{v}$.

Fast Fourier Transform (FFT)

- Consider $\mathbf{b} = \mathbf{F}\mathbf{a}$, we have

$$\forall j \in [0, n-1] \quad b_j = \sum_{k=0}^{n-1} \omega_{(n)}^{jk} a_k,$$

the FFT computes this recursively via 2 FFTs of dimension $n/2$, using $\omega_{(n/2)} = \omega_{(n)}^2$,

$$\begin{aligned} b_j &= \sum_{k=0}^{n/2-1} \omega_{(n)}^{j(2k)} a_{2k} + \sum_{k=0}^{n/2-1} \omega_{(n)}^{j(2k+1)} a_{2k+1} \\ &= \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k} + \omega_{(n)}^j \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1} \end{aligned}$$

Fast Fourier Transform Derivation

- The FFT leverages similarity between the first and second half of the output,

$$b_j = \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k}}_{u_j} + \omega_{(n)}^j \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1}}_{v_j}$$

corresponds closely to the entry shifted by $n/2$,

$$b_{j+n/2} = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{(j+n/2)k} a_{2k} + \omega_{(n)}^{j+n/2} \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{(j+n/2)k} a_{2k+1}$$

Now $\omega_{(n/2)}^{(j+n/2)k} = \omega_{(n/2)}^{jk}$ since $(\omega_{(n/2)}^{n/2})^k = 1^k = 1$ and using $\omega_{(n)}^{n/2} = -1$,

$$b_{j+n/2} = \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k}}_{u_j} - \omega_{(n)}^j \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1}}_{v_j}$$

FFT Algorithm Summary

- ▶ Let vectors $\mathbf{u}$ and $\mathbf{v}$ be two recursive FFTs, $\forall j \in [0, n/2 - 1]$

$$u_j = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k}, \quad v_j = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1}$$

- ▶ Given $\mathbf{u}$ and $\mathbf{v}$ scale using "twiddle factors" $z_j = \omega_{(n)}^j \cdot v_j$
- ▶ Then it suffices to combine the vectors as follows $\mathbf{b} = \begin{bmatrix} \mathbf{u} + \mathbf{z} \\ \mathbf{u} - \mathbf{z} \end{bmatrix}$
- ▶ The FFT has $O(n \log n)$ cost complexity:
There are two recursive calls of dimension $n/2$ and $O(n)$ work for application to twiddle factors and final summation, thus

$$T(n) = 2T(n/2) + O(n) = O(n \log n).$$

Applications of the FFT

- ▶ We can rapidly multiply degree n polynomials by considering their values $\omega_{(2n-1)}^i$ for $i \in \{0, \dots, 2n-1\}$

$$p_c(\omega_{(2n-1)}^i) = p_a(\omega_{(2n-1)}^i)p_b(\omega_{(2n-1)}^i)$$

- ▶ *The product of coefficients of p_a, p_b with Vandermonde matrix $v_{ij} = (\omega_{(2n-1)}^i)^j$, which is the DFT matrix, gives values of polynomials at $2n-1$ nodes.*
- ▶ *Interpolation to compute coefficients of p_c from the products of values of p_a and p_b at those nodes is multiplication by the inverted DFT matrix and is exact since p_c is degree $2n-2$.*
- ▶ More generally the DFT can be used to solve any Toeplitz linear system (convolution):
 - ▶ *A standard convolution has the form, $\forall k \in [0, n-1] \quad c_k = \sum_{j=0}^k a_j b_{k-j}$.*
 - ▶ *Convolution is equivalent to multiplications of polynomials with degree $n/2-1$ and coefficients a and b , where the convolution computes the coefficients c of the product of the two polynomials.*

Convolution via DFT

- ▶ The Fourier transform method for computing a convolution is given by

$$c_k = \frac{1}{n} \sum_s \omega_{(n)}^{-ks} \left(\sum_j \omega_{(n)}^{sj} a_j \right) \left(\sum_t \omega_{(n)}^{st} b_t \right)$$

- ▶ *Rearrange the order of the summations to see what happens to every product of a and b*

$$c_k = \frac{1}{n} \sum_s \sum_j \sum_t \omega_{(n)}^{(j+t-k)s} a_j b_t$$

- ▶ *For any $u = j + t - k \neq 0$, we observe $\sum_s (\omega_{(n)}^u)^s = 0$*
- ▶ *When $j + t - k = 0$ the products $\omega_{(n)}^{(s+t-j)k} = 1$, so there are n nonzero terms $a_j b_{k-j}$ in the summation*

Solving Numerical PDEs with the FFT

- ▶ 1D finite-difference schemes on a regular grid correspond to convolutions:
1D model problem is simply convolution with vector $[1, -2, 1]$.
- ▶ For the 1D Poisson model problem, the eigenvectors of T corresponds to the imaginary part of a minor of a $2(n+1)$ -dimensional DFT matrix:
 - ▶ *In particular, $T = XDX^{-1}$ where x_{ij} is the imaginary part of $f_{i+1,j+1}$ with $X \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{2(n+1) \times 2(n+1)}$.*
 - ▶ *Consequently, T can be diagonalized and the overall system solved by FFT with $O(n \log n)$ cost.*
- ▶ Multidimensional Poisson can be handled with multidimensional FFT:
For example 2D FFT (1D FFT of each row then 1D FFT of each column) suffices to solve the 2D Poisson problem.