

CS 450: Numerical Analysis¹

Interpolation

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¹*These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath ([slides](#)).*

Interpolation

- ▶ Given $(t_1, y_1), \dots, (t_m, y_m)$ with *nodes* $t_1 < \dots < t_m$ an *interpolant* f satisfies:

- ▶ Interpolant is usually constructed as linear combinations of *basis functions* $\{\phi_j\}_{j=1}^n = \phi_1, \dots, \phi_n$ so $f(t) = \sum_j x_j \phi_j(t)$.

Conditioning of Interpolation

- ▶ Conditioning of interpolation matrix A depends on basis functions and coordinates t_1, \dots, t_m :

- ▶ The Vandermonde matrix tends to be ill-conditioned:

Newton Basis

- ▶ The *Newton basis* functions $\phi_j(t) = \prod_{k=1}^{j-1} (t - t_k)$ with $\phi_1(t) = 1$ seek the best of monomial and Lagrange bases:

- ▶ The Newton basis yields a triangular Vandermonde system:

Orthogonal Polynomials

- ▶ Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of *orthonormal functions*:

Legendre Polynomials

- ▶ The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:

- ▶ The *Legendre polynomials* are obtained by Gram-Schmidt on the monomial basis, with $w(t) = \begin{cases} 1 : -1 \leq t \leq 1 \\ 0 : \text{otherwise} \end{cases}$ and normalized so $\hat{\phi}_i(1) = 1$.

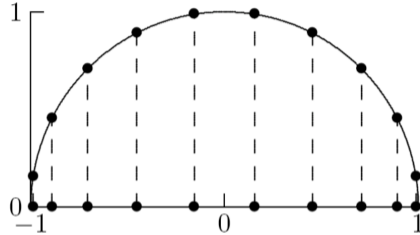
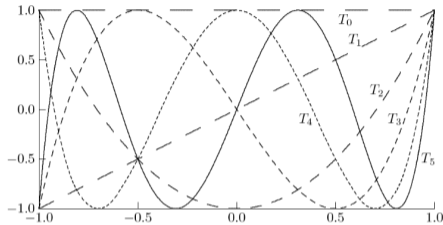
Chebyshev Basis

Demo: Chebyshev interpolation

Activity: Chebyshev Interpolation

- ▶ *Chebyshev polynomials* $\phi_j(t) = \cos((j - 1) \arccos(t))$ and *Chebyshev nodes* $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ provide a way to pick *nodes* t_1, \dots, t_n along with a basis, to yield perfect conditioning:

Chebyshev Nodes Intuition



- ▶ Note *equi-oscillation* property, successive extrema of $T_k = \phi_k$ have the same magnitude but opposite sign.
- ▶ Set of k Chebyshev nodes are given by zeros of T_{k+1} and are abscissas of points uniformly spaced on the unit circle.

Error in Interpolation

We show by induction that given degree n polynomial interpolant \tilde{f} of f the error $E(t) = f(t) - \tilde{f}(t)$ has n zeros t_1, \dots, t_n and there exist $y_1, \dots, y_n \in [t_1, t_n]$ so

Interpolation Error Bounds

- ▶ Consequently, polynomial interpolation satisfies the following error bound:

- ▶ Letting $h = t_n - t_1$ (often also achieve same for h as the node-spacing $t_{i+1} - t_i$), we obtain

Piecewise Polynomial Interpolation

- ▶ The k th piece of the interpolant is typically chosen as polynomial on $[t_i, t_{i+1}]$

- ▶ *Hermite* interpolation ensures consecutive interpolant pieces have same derivative at each *knot* t_i :

Spline Interpolation

- ▶ A *spline* is a $(k - 1)$ -time differentiable piecewise polynomial of degree k :

- ▶ The resulting interpolant coefficients are again determined by an appropriate *generalized Vandermonde system*:

B-Splines

B-splines provide an effective way of constructing splines from a basis:

- ▶ The basis functions can be defined recursively with respect to degree:
- ▶ ϕ_i^1 is a linear hat function that increases from 0 to 1 on $[t_i, t_{i+1}]$ and decreases from 1 to 0 on $[t_{i+1}, t_{i+2}]$.
- ▶ ϕ_i^k is positive on $[t_i, t_{i+k+1}]$ and zero elsewhere.
- ▶ The B-spline basis spans all possible splines of degree k with nodes $\{t_i\}_{i=1}^n$.
- ▶ The B-spline basis coefficients are determined by a Vandermonde system that is lower-triangular and banded (has k subdiagonals), and need not contain differentiability constraints, since $f(t)$ is a sum of ϕ_i^k s.