CS 450: Numerical Analysis
Initial Value Problems for Ordinary Differential Equations

University of Illinois at Urbana-Champaign

¹These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath (slides).
Ordinary Differential Equations

- An ordinary differential equation (ODE) usually describes time-varying system by a function $y(t)$ that satisfies a set of equations in its derivatives.

- An ODE of any order $k$ can be transformed into a first-order ODE,
Example: Newton’s Second Law

▶ Consider, \( F = ma \) for a given force \( F \), which is a second order ODE,

▶ We can transform it into a first order ODE in two variables:
Initial Value Problems

- Generally, a first order ODE specifies only the derivative, so the solutions are non-unique. An *initial condition* addresses this:

- Given an initial condition, an ODE must satisfy an integral equation for any given point $t$: 
Existence and Uniqueness of Solutions

- For an ODE to have a unique solution, it must be defined on a closed domain $D$ and be *Lipschitz continuous*:

- The solutions of an ODE can be stable, unstable, or asymptotically stable:
Stability of 1D ODEs

The solution to the scalar ODE \( \frac{dy}{dt} = \lambda y \) is \( y(t) = y_0 e^{\lambda t} \), with stability dependent on \( \lambda \):

A constant-coefficient linear ODE has the form \( \frac{dy}{dt} = Ay \), with stability dependent on the real parts of the eigenvalues of \( A \):
Numerical Solutions to ODEs

Methods for numerical ODEs seek to approximate $y(t)$ at $\{t_k\}_{k=1}^m$.

Euler’s method provides the simplest method (attempt) for obtaining a numerical solution:
Error in Numerical Methods for ODEs

- Truncation error is typically the main quantity of interest, which can be defined \textit{globally} or \textit{locally}:

- The \textit{order of accuracy} of a given method is one less than than the order of the leading order term in the local error $l_k$:
Accuracy and Taylor Series Methods

- By taking a degree-$r$ Taylor expansion of the ODE in $t$, at each consecutive $(t_k, y_k)$, we achieve $r$th order accuracy.

- Taylor series methods require high-order derivatives at each step:
Growth Factors and Stability Regions

- Stability of an ODE method discerns whether local errors are amplified, deamplified, or stay constant:

- Basic stability properties follow from analysis of linear scalar ODE, which serves as a local approximation to more complex ODEs.
The stability region of a general ODE constrains the eigenvalues of $hJ_f$. 
Implicit methods for ODEs form a sequence of solutions that satisfy conditions on a local approximation to the solution:

The stability region of the backward Euler method is the left half of the complex plane:
Stiffness

- *Stiff* ODEs are ones that contain components that vary at disparate time-scales:
A second-order accurate implicit method is the trapezoid method.

Generally, methods can be derived from quadrature rules:
Multi-Stage Methods

▶ **Multi-stage methods** construct $y_{k+1}$ by approximating $y$ between $t_k$ and $t_{k+1}$:

The 4th order Runge-Kutta scheme is particularly popular:

*This scheme uses Simpson’s rule,*

$$
y_{k+1} = y_k + (h/6)(v_1 + 2v_2 + 2v_3 + v_4)
$$

$$
v_1 = f(t_k, y_k), \quad v_2 = f(t_k + h/2, y_k + (h/2)v_1),
$$

$$
v_3 = f(t_k + h/2, y_k + (h/2)v_2), \quad v_4 = f(t_k + h, y_k + hv_3).
$$
Runge-Kutta Methods

- Runge-Kutta methods evaluate $f$ at $t_k + c_i h$ for $c_0, \ldots, c_r \in [0, 1]$.

- A general family of Runge Kutta methods can be defined by
Multistep Methods

- **Multistep methods** employ \( \{y_i\}_{i=0}^{k} \) to compute \( y_{k+1} \):

- Multistep methods are not self-starting, but have practical advantages: