

CS 450: Numerical Analysis¹

Boundary Value Problems for Ordinary Differential Equations

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¹*These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath ([slides](#)).*

Boundary Conditions

- ▶ Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary.

- ▶ Consider a first order ODE $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y})$ with *linear boundary conditions* on domain $t \in [a, b]$:

$$\mathbf{B}_a \mathbf{y}(a) + \mathbf{B}_b \mathbf{y}(b) = \mathbf{c}$$

Existence of Solutions for Linear ODE BVPs

- ▶ The solutions of linear ODE BVP $\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{b}(t)$ are linear combinations of solutions to linear homogeneous ODE IVPs $\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t)$:

- ▶ Solution $\mathbf{u}(t)$ (and $\mathbf{y}(t)$) exists if $\mathbf{Q} = \mathbf{B}_a\mathbf{Y}(a) + \mathbf{B}_b\mathbf{Y}(b)$ is invertible:

Green's Function Form of Solution for Linear ODE BVPs

- ▶ For any given $b(t)$ and c , the solution to the BVP can be written in the form:

$$\mathbf{y}(t) = \Phi(t)\mathbf{c} + \int_a^b \mathbf{G}(t, s)\mathbf{b}(s)ds$$

$\Phi(t) = \mathbf{Y}(t)\mathbf{Q}^{-1}$ is the *fundamental matrix* and the *Green's function* is

$$\mathbf{G}(t, s) = \mathbf{Y}(t)\mathbf{Q}^{-1}\mathbf{I}(s)\mathbf{Y}^{-1}(s), \quad \mathbf{I}(s) = \begin{cases} \mathbf{B}_a\mathbf{Y}(a) & : s < t \\ -\mathbf{B}_b\mathbf{Y}(b) & : s \geq t \end{cases}$$

Conditioning of Linear ODE BVPs

- ▶ For any given $\mathbf{b}(t)$ and \mathbf{c} , the solution to the BVP can be written in the form:

$$\mathbf{y}(t) = \mathbf{\Phi}(t)\mathbf{c} + \int_a^b \mathbf{G}(t,s)\mathbf{b}(s)ds$$

- ▶ The absolute condition number of the BVP is $\kappa = \max\{\|\mathbf{\Phi}\|_\infty, \|\mathbf{G}\|_\infty\}$:

Shooting Method for ODE BVPs

- ▶ For linear ODEs, we construct solutions from IVP solutions in $\mathbf{Y}(t)$, which suggests the *shooting method* for solving BVPs by reduction to IVPs:

- ▶ *Multiple shooting* employs the shooting method over subdomains:

Finite Difference Methods

- ▶ Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

- ▶ Convergence to solution is obtained with decreasing step size h so long as the method is consistent and stable:

Finite Difference Methods

- ▶ Lets derive the finite difference method for the ODE BVP defined by

$$u'' + 7(1 + t^2)u = 0$$

with boundary conditions $u(-1) = 3$ and $u(1) = -3$, using a centered difference approximation for u'' on t_1, \dots, t_n , $t_{i+1} - t_i = h$.

Collocation Methods

- ▶ *Collocation methods* approximate \mathbf{y} by representing it in a basis

$$\mathbf{y}(t) \approx \mathbf{v}(t, \mathbf{x}) = \sum_{i=1}^n x_i \phi_i(t).$$

- ▶ Choices of basis functions give different families of methods:

Solving BVPs by Optimization

- ▶ To improve robustness, define and minimize a residual error over the whole domain rather than at collocation points.

- ▶ The first-order optimality conditions of the optimization problem are a system of linear equations $\mathbf{Ax} = \mathbf{b}$:

Weighted Residual

- ▶ *Weighted residual methods* work by ensuring the residual is orthogonal with respect to a given set of weight functions:

- ▶ The *Galerkin method* is a weighted residual method where $w_i = \phi_i$.

Second-Order BVPs: Poisson Equation

In practice, BVPs are at least second order and its advantageous to work in the natural set of variables.

- ▶ Consider the *Poisson equation* $u''(t) = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:

- ▶ Defining residual equation by analogy to the first order case, we obtain,

Weak Form and the Finite Element Method

- ▶ The finite-element method permits a lesser degree of differentiability of basis functions by casting ODEs such as Poisson in *weak form*:

Eigenvalue Problems with ODEs

- ▶ A typical second-order scalar *ODE BVP eigenvalue problem* is to find eigenvalue λ and eigenfunction u to satisfy

$$u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0.$$

These can be solved, e.g. for $f(t, u, u') = g(t)u$ by finite differences:

Using Generalized Matrix Eigenvalue Problems

- ▶ Generalized matrix eigenvalue problems arise from more sophisticated ODEs,

$$u'' = \lambda(g(t)u + h(t)u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0.$$