CS 450: Numerical Anlaysis[1]

Fast Fourier Transform

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[1] *These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book "Scientific Computing: An Introductory Survey" by Michael T. Heath (slides).*

Sparse Linear Systems and Time-independent PDEs

- The Poisson equation serves as a model problem for numerical methods:
- Dense, sparse direct, iterative, FFT, and Multigrid methods provide increasingly good complexity for the problem:

Multigrid

- Multigrid employs a hierarchy of grids to accelerate iterative methods:

- The multigrid method works by resolving high-frequency error components on finer-grids and low-frequency error components on coarser grids:

Multigrid

- Consider the Galerkin approximation with linear finite elements to the Poisson equation $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$:

$$\phi_i^{(h)}(t) = \begin{cases} (t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\ (t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\ 0 & : \text{otherwise} \end{cases}$$

where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$.

Coarse Grid Matrix

- Multigrid restricts the residual equation on the fine grid $\boldsymbol{A}^{(h)}\boldsymbol{x} = \boldsymbol{r}^{(h)}$ to the coarse grid:

Restricting the Residual Equation

- Given the fine-grid residual $\boldsymbol{r}^{(h)}$, we seek to use the coarse grid to approximate $\boldsymbol{x}^{(h)}$ so that $\boldsymbol{A}\boldsymbol{x}^{(h)} \approx \boldsymbol{r}^{(h)}$

Discrete Fourier Transform

- The solutions to hyperbolic PDEs like Poisson are wave-like and take on simple representations in the frequency basis, both for continuous and discretized equations. We define the *discrete Fourier transform* using

$$\omega_{(n)} = \cos(2\pi/n) - i\sin(2\pi/n) = e^{-2\pi i/n}.$$

Fast Fourier Transform (FFT)

- Consider $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{a}$, we have

$$\forall j \in [0, n-1] \quad b_j = \sum_{k=0}^{n-1} \omega_{(n)}^{jk} a_k,$$

  the FFT computes this recursively via 2 FFTs of dimension $n/2$, using $\omega_{(n/2)} = \omega_{(n)}^2$,

Fast Fourier Transform Derivation

- The FFT leverages similarity between the first and second half of the output,

$$b_j = \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k}}_{u_j} + \omega_{(n)}^{j} \underbrace{\sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1}}_{v_j}$$

  corresponds closely to the entry shifted by $n/2$,

$$b_{j+n/2} = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{(j+n/2)k} a_{2k} + \omega_{(n)}^{j+n/2} \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{(j+n/2)k} a_{2k+1}$$

FFT Algorithm Summary

- Let vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ be two recursive FFTs, $\forall j \in [0, n/2-1]$

$$u_j = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k}, \quad v_j = \sum_{k=0}^{n/2-1} \omega_{(n/2)}^{jk} a_{2k+1}$$

- The FFT has $O(n \log n)$ cost complexity:

Applications of the FFT

- We can rapidly multiply degree n polynomials by considering their values $\omega^i_{(2n-1)}$ for $i \in \{0, \ldots, 2n-1\}$

- More generally the DFT can be used to solve any Toeplitz linear system (convolution):

Convolution via DFT

- The Fourier transform method for computing a convolution is given by

$$c_k = \frac{1}{n} \sum_s \omega_{(n)}^{-ks} \Big(\sum_j \omega_{(n)}^{sj} a_j \Big) \Big(\sum_t \omega_{(n)}^{st} b_t \Big)$$

Solving Numerical PDEs with the FFT

- 1D finite-difference schemes on a regular grid correspond to convolutions:

- For the 1D Poisson model problem, the eigenvectors of $\boldsymbol{T}$ corresponds to the imaginary part of a minor of a $2(n+1)$-dimensional DFT matrix:

- Multidimensional Poisson can be handled with multidimensional FFT: