April 20, 2026

Announcements

- exam 4

  retake next week

- $\vec{x} = [\;]$ $\quad$ $\vec{y} = [\;]$

  $\vec{x}^T\vec{y}$ ← inner

  $\vec{x}\vec{y}^T$ ← outer

Goals

- IVP theory ✓

  ∃, Unique, Sensitivity

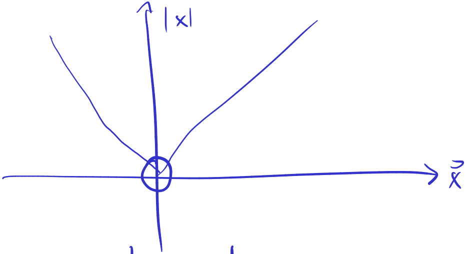
- Examples

- Euler methods

- Accuracy / Stability ✓

Lipschitz continuity

$$\|f(\vec{y}) - f(\tilde{\vec{y}})\| \le L \, \|\vec{y} - \tilde{\vec{y}}\|$$



$z' = \lambda z$

$z(t) = e^{\lambda t}$

$\vec{y}\,' = f(\vec{y})$

$\vec{y}(0) = \vec{y}_0$ or $\tilde{\vec{y}}_0$

$$\|\tilde{\vec{y}}(t) - \vec{y}(t)\| \le e^{Lt} \, \|\tilde{\vec{y}}_0 - \vec{y}_0\|$$

$\varepsilon - \delta$

For all $\varepsilon > 0$

there exists a $\delta > 0$

so that if

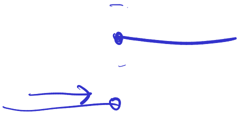
$\cancel{|x - \tilde{x}| < \delta}$ $\|y_0 - \tilde{y}_0\| < \delta$

then

$\cancel{|f(x) - f(\tilde{x})|} < \varepsilon$

$\|\tilde{y} - y\|_\infty$

t



Asymptotic stability

$$y' = \lambda y$$

$$\vec{y}' = A\vec{y}$$

$$A = V D V^{-1}$$

$$\vec{z} := V^{-1}\vec{y} \iff \vec{y} = V\vec{z}$$

$$(V\vec{z})' = A V \vec{z}$$

$$= V D V^{-1} V \vec{z}$$

$$\cancel{V}\vec{z}' = \cancel{V} D \vec{z}$$

Methods

$$y' = f(y) \iff y(t) = y_0 + \int_0^t f(y(t))\,dt$$
$$y(0) = y_0$$

$$= y_0 + \sum w_i f(y(t_i))\,dt$$

"Left endpoint"

$$y(\Delta t) = y_0 + \int_0^{\Delta t} f(y(t))\,dt$$

$$\approx y_0 + \Delta t \cdot f(y_0)$$

"Forward Euler"/explicit

"Right endpoint"

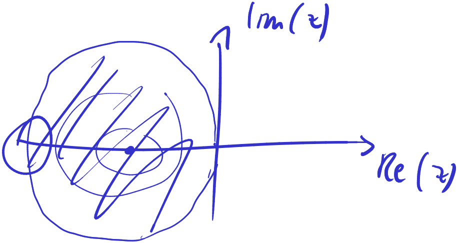
$$\underline{y(\Delta t)} \approx y_0 + \Delta t \cdot f(\underline{y(\Delta t)})$$

"Backward Euler"/ implicit

$$y_{k+1} = y_k + \Delta t\, f(y_k)$$

$$y' = \lambda y$$

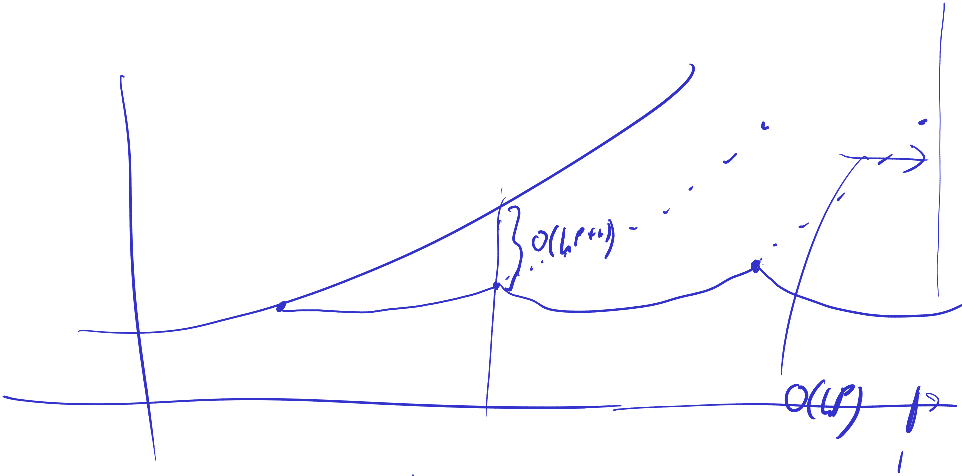
$$y_{k+1} = y_k + \Delta t\, \lambda\, y_k$$
$$= (1 + \Delta t\, \lambda)\, y_k$$
$$= (1 + \Delta t\, \lambda)^{k+1}\, y_0$$



$$|1 + \underbrace{\Delta t\, \lambda}_{z \in \mathbb{C}}| < 1$$

$$R(z) = 1 + z$$

$$z = \Delta t\, \lambda$$



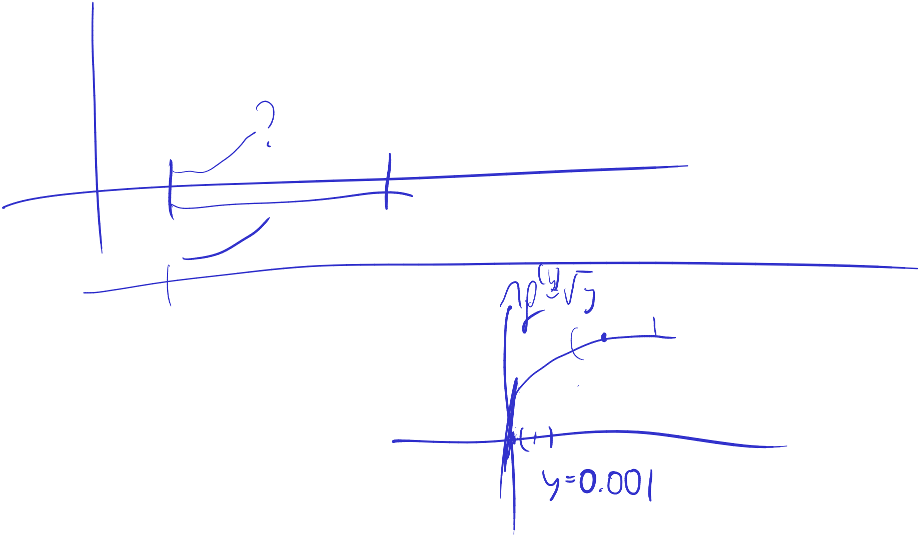
$h \to 0$

$\xi \in (0,h)$

$$f(x+h) = f(x) + f'(\xi)h$$

$$f(x+h) - f(x) = \underline{f'(\xi)}h$$

$$|f(x+h) - f(x)| \leq L h$$


?
y=0.001