April 27, 2026

Announcements

- 4CH 2
- Exam 4 grades
  - Retake
- HW 8 out

Goals

BVPs

- Why?
- Solution theory
- Methods
  - Shooting
  - FD
    ↳ Sparse matrices
  - Weighted residuals

BVPs: why?

- Structural

Dirichlet

$$u'' = f$$

- Heat conduction

heat equation

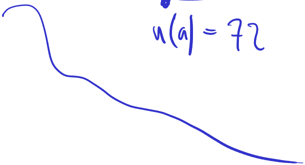
0

$$\cancel{u_t} = u_{xx}$$

$$\frac{\partial}{\partial t}u \qquad \frac{\partial^2}{\partial x^2}$$

$u(a) = 72$ $\qquad$ $u'(b) = 0$

- Pollution



- Separation of variables $u(x,y) = U(x)\,V(y)$

Solving

$$\vec{y}\,' = \cancel{f(y(t))}$$

nonlinear

$$\vec{y}\,'(x) = A(x)\,\vec{y}(x) + \vec{b}(x)$$

$$B_a\,\vec{y}(a) + B_b\,\vec{y}(b) = \vec{c}$$

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$\ominus$

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$$B_a\,\vec{y}(a) + B_b\,\vec{y}(b) = \vec{0}$$

For system in $\mathbb{R}^n$ expect n lin. indep solutions

↪ n BCs to resolve the ambiguity.

$$Y'(x) = A(x)\,Y(x)$$ ← obeys solution linearity.

first col: $Y(a) = \vec{e}_1$ . . . $\vec{y}$ and $\vec{w}$

$\alpha\vec{y}$

$$\underbrace{z(x) = \phi(x)\,\vec{c}}$$

$$B\,\vec{z}(a) + B\,\vec{z}(b) = \vec{c}$$

Vol-inh. BVP:

$$\vec{y}'(x) = A(x)\,\vec{y}(x) + \vec{b}(x)$$

$$B\,\vec{y}(a) + B\,\vec{y}(b) = \vec{0}$$

$$\vec{y} = A\vec{b}$$

$$y_v(x) = \int_a^b G(x,z)\,b(z)\,dz$$

$$y_i = \sum_j A_{ij} b_j$$

"one column": $G(\cdot, z)$

Criteria for G:

- ODE
- obey (homogeneous) BCs
- Jump condition: $G(z^+, z) - G(z^-, z) = I$

$$y_v(x) = \int_a^b G(x,z)\, b(z)\, dz$$

$$= \int_a^x G(x,z)\, b(z)\, dz + \int_x^b G(x,z)\, b(z)\, dz$$

$$y_v'(x) = \underbrace{\int_a^x \frac{\partial}{\partial x} G(x,z)\, b(z)\, dz + \ldots}_{\text{ODE}} + G(x,x^-)\, b(x) \quad G(x,x^+)\, b(x)$$

Methods

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

CSR sparse matrices

row(i) : [indptr(i) .. indptr[i+1])

Neuman BC

$$u'(x) = \alpha u(x) + \beta u(x+h) + \gamma u(x+2h)$$

Weighted Residual

$$u''(x) = f(x) \qquad u(a) = u(b) = 0$$

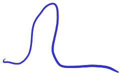
$$\int u''(x)\,\psi(x)\,dx = \int f(x)\,\psi(x)\,dx$$

for all $\psi \in$?

test

$$\psi(x) = \delta(x-c)$$

↳ gives ODE at a point c



$$\int_a^b u''(x)\,\psi(x)\,dx = \Big[u'(x)\,\psi(x)\Big]_a^b - \underbrace{\int u'(x)\,\psi'(x)\,dx}_{\text{finite element}}$$

maybe choose test f, so $= 0$