March 23, 2026

Announcements

HW5

bugs

Exam 3

next week

Goals

Optim.

Existence

Uniqueness

Sensitivity

Optimality condition

GSS

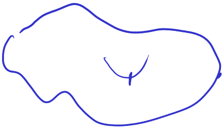
Newton

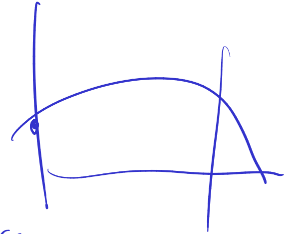
Existence

$f: \mathbb{R}^n \to \mathbb{R}$

- $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ $\Rightarrow \exists$ minimum

  S closed bounded

S 

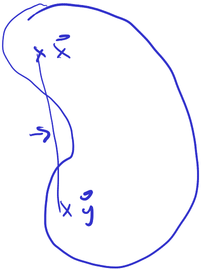


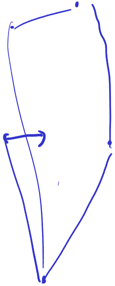
- coercive $f \to \infty$ as $\|x\| \to \infty$

  $\forall E > 0 \ \exists d > 0 : \|\vec{x}\| > d \Rightarrow f(\vec{x}) > E$

Uniqueness: convexity

sets



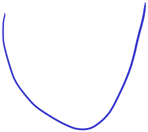


$\alpha \tilde{x} + (1-\alpha)\tilde{y}$

convex combination

$\sum_{i=1}^{5} \alpha_i \tilde{x}_i$ $\quad \sum_{i=1}^{5} \alpha_i = 1 \quad \alpha_i \geq 0$

functions

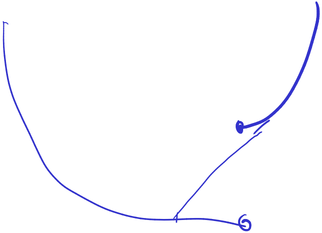


$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$

$<$ ← strict

c: local min ⇒ global min

sc: local min ⇒ unique global min



nD Taylor

$f: \mathbb{R}^n \to \mathbb{R}$

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + \nabla f(x) \cdot \vec{h} + \frac{1}{2} \vec{h}^T H_f(x) \vec{h}$$

$H_f = V D V^T$ $\quad V^{-1} = V^T$ because V orth.

added after class! ⚠

$$\vec{h}^T H_f \vec{h} = \vec{h}^T (V D V^T) \vec{h} = \underbrace{(V^T \vec{h})^T}_{\vec{y}} D (V^T \vec{h})$$

$$= \vec{y}^T D \vec{y} = \sum_{i=1}^{n} d_{ii} y_i^2$$

$\|y\|_2 = \|\vec{h}\|_2$

Optimality crit (uD)

$\nabla f(x^*) = 0$: necessary

$H_f(x^*)$ pos. definite

Sensitivity (nD)

$$f(\vec{x}^* + h\vec{s}) = f(x^*) + \underbrace{\cancel{\nabla f(x^*)^T h\vec{s}}}_{0} + \frac{h^2}{2} s^T H_f(\vec{x}^*)\vec{s} + O(h^3)$$

$\|s\| = 1$

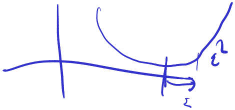
$$|f(\tilde{x}) - f(x^*)| \le tol$$

$\uparrow$ max $\quad \downarrow$ min

$$\Rightarrow \underbrace{|f(\tilde{x}) - f(x^*)|}_{tol \,\ge} = |f(\vec{x}^* + h\vec{s}) - f(x^*)| = \frac{h^2}{2}|s^T H_f(x^*)\, s|$$

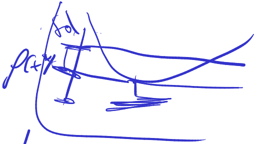
$\tilde{x} = x^* + hs$

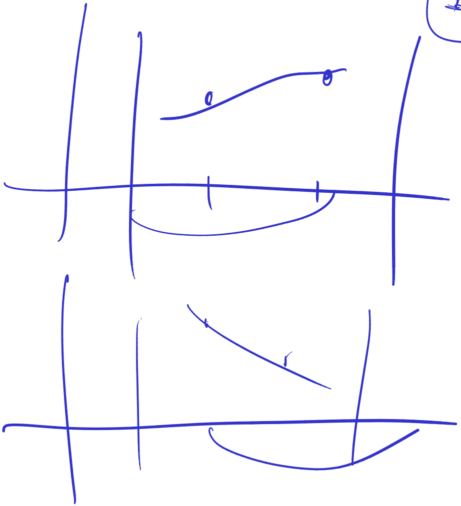
$$|h|^2 \le \frac{2tol}{\lambda_{min}} \Leftrightarrow h \le \sqrt{\frac{2tol}{\lambda_{min}}}$$

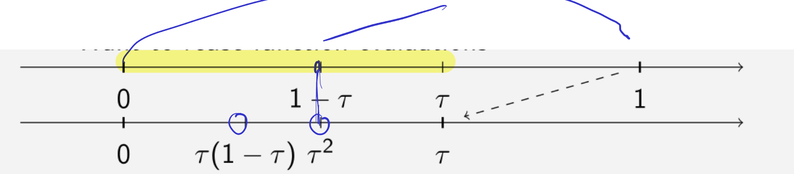


Bisection

Unimodality





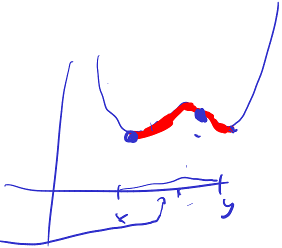


$$\tau^2 = 1 - \tau$$

Newton for opt; use solve Newton on $f' = 0$

$$x_{k+1} = x_k - \frac{f'}{f''}$$

$$\alpha \vec{x} + (1-\alpha)\, \vec{y} \qquad \alpha \in [0,1]$$



$$f(\alpha x + (1-\alpha) y) \underset{<}{\leq} \alpha f(x) + (1-\alpha) f(y)$$

For all $\alpha \in [0,1]$