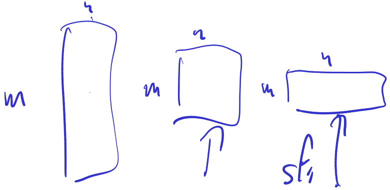
February 11, 2026

Announcements

- HW2
- Exam graded Friday

Goals

- Normal equations
- Lsq and orthogonality
- LA review: projectors
- $\underbrace{\text{full-and-skinny}}$ cond #



LS:

square may be invertible

$\exists$ nullsp!

"full rank": $\operatorname{rank}(A) = \min(m, n)$

rank - nullity theorem:

$$\dim N(A) + \underbrace{\dim \operatorname{colspan}(A)}_{\operatorname{rank}(A)} = \# \text{cols}$$

Least squares: short - fat

$$\tilde{x} = \arg\min \| A\tilde{x} - \tilde{b} \|_2^2 \iff A\tilde{x} \overset{\sim}{=} \tilde{b}$$

$$\varphi(x) = a + bx + cx^2 = y$$

$$(x_i, y_i)$$

$$\tilde{y} = \varphi(\tilde{x}) \hat{=} a \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + b \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + c \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & x_1 & x_1^2 \\ \vdots & & \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Vandermonde

$\psi_1 \cdots \psi_n$:

generalized Vandermonde

$$\vec{y} \simeq \begin{bmatrix} \psi_1(x_1) & \psi_k(x_1) \\ \vdots & \\ \psi_1(x_n) & \psi_k(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_n \end{bmatrix}$$

$$\varphi(x) = a_1 \psi_1(x) + \cdots + a_n \psi_n(x)$$

Linearity: $f(\alpha \vec{a} + \beta \vec{b}) = \alpha f(\vec{a}) + \beta f(\vec{b})$

$$\varphi(x) = a_1 \varphi_1(x) + \cdots + a_n \varphi_n(x)$$

$$\gamma(\tilde{x}) = \| \tilde{b} - Ax \|_2^2$$

$$= (\tilde{b} - A\tilde{x})^T (\tilde{b} - A\tilde{x})$$

$$= b^T b - 2\, x^T A^T b + x^T A^T A \tilde{x}$$

$$\nabla_x \gamma(x) = 2 A^T \tilde{b} - A^T A \tilde{x}$$

$$\vec{x}^T A^T A \vec{x} = \sum_i \left(\sum_j A_{ij} x_j \right)^2$$

$$\partial_{x_k} \vec{x}^T A^T A \vec{x} = \sum_i 2 \left[\partial_{x_k} \left(\sum_j A_{ij} x_j \right) \right] \left(\sum_j A_{ij} x_j \right)$$

$$= \sum_i 2 \left[\left(\sum_j A_{ij} \delta_{jk} \right) \right] \left(\sum_j A_{ij} x_j \right)$$

$$= \sum_i 2 A_{ik} \sum_j A_{ij} x_j = \left[2 A^T A \vec{x} \right]_k$$

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & \text{otherwise} \end{cases}$$

"Kronecker delta"

$$\mathrm{cond}(AB) = \|AB\| \, \|(AB)^{-1}\|$$

$$= \|AB\| \, \|B^{-1}A^{-1}\|$$

$$\leq \|A\| \, \|A^{-1}\| \, \|B\| \, \|B^{-1}\|$$

$$= \mathrm{cond}(A) \, \mathrm{cond}(B)$$

added later!

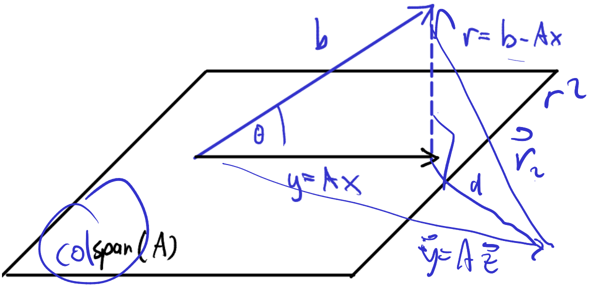
$$\|A^T\|_2 = \sigma_{max}(A^T)$$

$$= \sigma_{max}(A) = \|A\|_2$$

$A = U \Sigma V^T$

$A^T = V \Sigma^T U^T$

$(A^{-1})^T = (A^T)^{-1} = A^{-T}$



$$\|\vec{r}_2\|_2^2 = d^2 + \|\vec{r}\|_2^2$$

$\min \|Ax - b\|_1 \to$ much harder

$$A^T A \vec{x} = A^T b \iff \vec{x} = \underbrace{(A^T A)^{-1} A^T b}$$

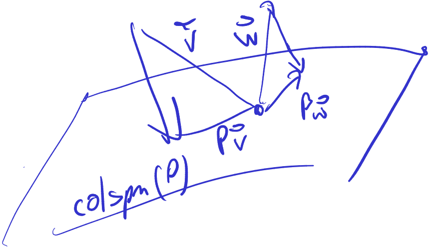
$$A^+ = (A^T A)^{-1} A^T$$ pseudo inverse

$$P_A = A A^+ = A (A^T A)^{-1} A^T$$

projector: $P^2 = P$

orthogonal projector: $P^2 = P$ and $P^T = P$

⚠ not an orthogonal matrix



$$\vec{v}^T P \vec{w} = (P\vec{v})^T \vec{w} = \vec{v}^T P^T \vec{w}$$

$$\iff P = P^T$$

$$\text{cond}(A) = \|A\| \, \|A^{-1}\| \quad \text{if invertible}$$

$\underline{\text{cond}(A) = \infty \text{ if}}$ A has a null space

$A\vec{x} = \vec{b}$

$$\frac{\|\Delta x\|}{\|x\|} \le \kappa \cancel{\frac{\|\Delta b\|}{\|b\|}}$$

$$\vec{x} + \alpha \vec{n}$$

$\vec{n} \in N(A)$

arbitrarily large

$\mathrm{cond}_2(A) = \|A\|_2 \|A^+\|_2$ no nullspace

$A\vec{x} = \vec{b}$

$\|b\| = \|A\vec{x}\|$

$\leq \|A\| \, \|\vec{x}\|$

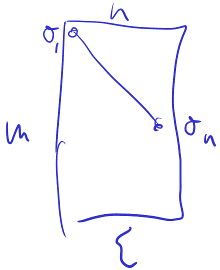
$$\frac{\frac{\|\Delta x\|}{\|x\|}}{\frac{\|\Delta b\|}{\|b\|}} = \frac{\|\Delta x\| \, \|b\|}{\|\Delta b\| \, \|x\|}$$

$\Delta\vec{x} = \cancel{A^{-1}} \Delta\vec{b}$

$\|\Delta x\|$

$\|A \Delta x\| = \|\Delta b\|$

$= \|U \Sigma V^T \Delta x\| = \|\Sigma \Delta x\|$



$A = \ldots$

$A\vec{x} \approx \vec{b}$

la.solve(A.T @ A, A.T @ b) $A^T A \vec{x} = A^T \vec{b}$

$$[A^T A]_{ij} = \sum_{k} [A^T]_{ik} [A]_{kj}$$

$$A = U \Sigma V^T$$

$$A^T = V \Sigma^T U^T$$