February 18, 2026

Announcements

- Feedback
- Exam 1 second-chance (graded Fri)
- HW2 due tonight
- HW3 comes out tomorrow

Goals

- Householder
- Givens
- Not-full-rank $\left\{ \begin{array}{l} \text{LSQ} \\ \text{QR} \end{array} \right.$

$A = QR$

Gram-Schmidt orthogonalizes

idea: get rid of dot products

U orthogonal: $U^T U = I$ $\quad U U^T = I$

$\|U\|_2 =$ max sing. val $= 1$

$U = U I I$ $\qquad U^{-1} = U^T$

$\mathrm{cond}_2(U) = \|U\|_2 \|U^{-1}\|_2 = \|U\|_2 \|U^T\|_2 = 1$

$$\underbrace{U_{17000} \cdots U_4 U_3 U_2 U_1}A = R$$

$$A = U_1^T \cdots U_{17000}^T R$$

Orthonormal basis:

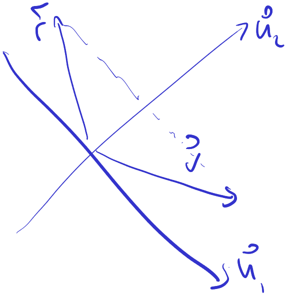
$$\vec{u}_1, \ldots, \vec{u}_n \qquad \vec{u}_i^T \vec{u}_j = \delta_{ij}$$

Let $\vec{v} \in \operatorname{span}(\vec{u}_1, \ldots, \vec{u}_n)$

$$\vec{v} = (\vec{u}_1^T \vec{v})\vec{u}_1 + (\vec{u}_2^T v)\vec{u}_2 + \cdots + (\vec{u}_n^T \vec{v})\vec{u}_n$$

$$-2(\vec{u}_1 \vec{u}_1^T)$$

$$\vec{r} = -(\vec{u}_1^T \vec{v})\vec{u}_1 + (\vec{u}_2^T \vec{v})\vec{u}_2 + \dots + (\vec{u}_n^T \vec{v})\vec{u}_n$$



$$H = I - 2\frac{\vec{v}\vec{v}^T}{v^T v}$$

Suppose we assume $\|\vec{v}\|_2 = 1$

$$H = I - 2\vec{v}\vec{v}^T$$

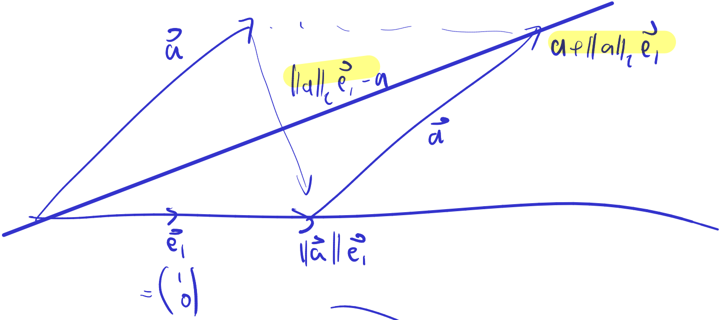
$$\|U\vec{x}\|_2 = \|x\|_2$$

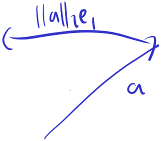
$$H \begin{bmatrix} \vec{a} & \cdots \end{bmatrix} = \begin{bmatrix} R_{11} & \cdots \\ 0 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} R_{11} \\ 0 \end{bmatrix} \to \|a\|_2 \, \vec{e}_1$$

$$R_{11} = \pm \|\vec{a}\|_2$$

$$\|H\vec{a}\|_2 = \|a\|_2$$





$$\vec{v} = \|a\|_2 \vec{e}_1 - \vec{a}$$

$$\vec{v} = \vec{a} \pm \|a\|_2 \vec{e}_1$$

Suppose $u_1 \ldots u_n$ form an ONB

$$(I\vec{x}) = (u_1^T x)u_1 + \cdots + (\vec{u}_n^T \vec{x})\,\vec{u}_n$$

$$\cdot(-1)$$

$$= -(u_1^T x)u_1 + \cdots + (\vec{u}_n^T \vec{x})\,\vec{u}_n$$

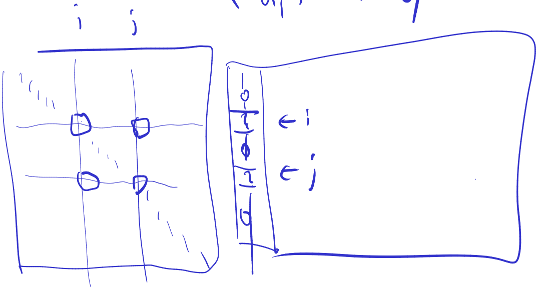
$$= I - 2(u_1^T x)u_1$$

$$H\vec{x} = I\vec{x} - \frac{2v(v^T \vec{x})}{v^T v}$$

Givens

$$\frac{1}{\|a\|_2}\overset{G}{\begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ? \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}^T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$



Rank-deficient.

LSQ

$\|Ax-b\|_2 \to \min!$

$\|x\|_2 \to \min!$

QR

zero diagonals

$PA=LU$

$AP=QR$

$H_1 A = \boxed{0 \;\; ?} = A_1$

$H_2 A_1 =$