

February 23, 2026

Announcements

- 1 ET
- Office hour shuffle
- HW3 typo fix
- Formula sheet for exam 2
- Exam 1 second page grades available

Goals

- LSQ
 - rank-deficient
 - SVD - fast
- Eigen values

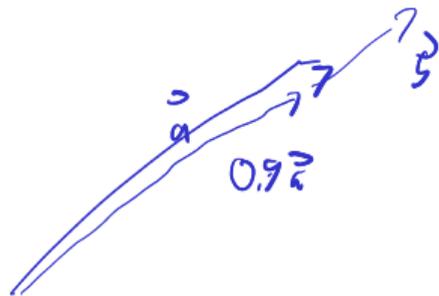
SVD

$$\begin{matrix} & n \\ m & \boxed{A} \\ & \end{matrix} = \begin{matrix} & n \\ m & \boxed{U} \\ & \end{matrix} \begin{matrix} & r \\ n & \boxed{\Sigma} \\ & \end{matrix} \begin{matrix} & r \\ n & \boxed{V^T} \\ & \end{matrix}$$

$$\begin{matrix} & n \\ m & \boxed{} \\ & \end{matrix} = \begin{matrix} & n \\ m & \boxed{U} \\ & \end{matrix} \begin{matrix} & m \\ m & \boxed{} \\ & \end{matrix} \begin{matrix} & r \\ m & \boxed{V^T} \\ & \end{matrix}$$

"economy / reduced"

Numerical rank



rank : not computable

num. rank $_{\epsilon}(A) = \# \{ \sigma_i \geq \epsilon : \sigma_i \text{ singular values of } A \}$

Low rank approx

$$A = \begin{array}{|c|c|c|c|} \hline \sigma_1 & 0 & & \\ \hline u_1 & u_2 & & \\ \hline \end{array}$$

n



$$\begin{array}{|c|c|c|c|} \hline v_1 & & & \\ \hline v_2 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

r^+

$$= \sum_{i=1}^r u_i v_i^T \sigma_i$$

Theorem (Eckart-Young-Mirsky)

If $k < r = \text{rank}(A)$ and

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^T, \text{ then}$$

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

$$\min_{\text{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k+1}^n \sigma_j^2}$$

Rank-deficient LSQ

$$A \vec{x} \approx \vec{b}$$



rank-deficient.

$$\begin{aligned} \|A \vec{x} - \vec{b}\|_2 &= \|U \Sigma V^T \vec{x} - \vec{b}\|_2 \\ &= \left\| \underbrace{\Sigma V^T \vec{x}}_{\vec{y}} - U^T \vec{b} \right\|_2 \\ &= \|\Sigma \vec{y} - U^T \vec{b}\|_2 \end{aligned}$$

$$\| \Sigma \| \quad \Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \dots \end{pmatrix} \quad \| \hat{y} - U^T b \|_2$$

$$V^T x = \hat{y}$$

$$y_i = \begin{cases} (U^T b)_i / \sigma_i & \text{if } \sigma_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & \\ & & \ddots & \\ & & & 1/\sigma_n & \\ & & & & & 0 \end{pmatrix} \quad A^+ = V \Sigma^+ U^T$$

Eigvalore

$$\vec{x}(t)$$

$$\vec{F}(t) = A \vec{x}(t)$$

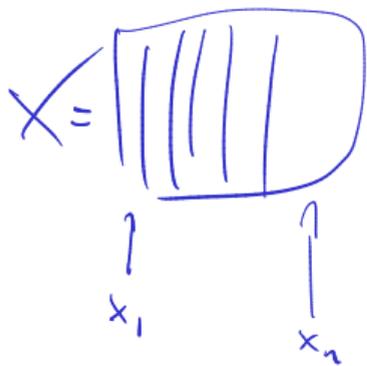
$$F = ma = m \frac{d^2 \vec{x}}{dt^2}$$

$$\frac{d^2 \vec{x}}{dt^2} = A \vec{x}(t)$$

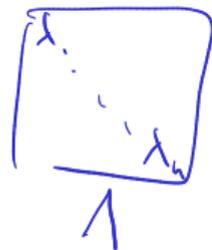
$$\vec{x}(t) = \vec{x}_0 \sin(\omega t)$$

$$-\omega^2 \vec{x}_0 \sin(\omega t) = A \vec{x}_0 \sin(\omega t)$$

"Spectral radius" = mg. of biggest eigenvalue



$$Ax_i = \lambda_i x_i \quad (\vec{x}_i \neq \vec{0})$$



$$AX = X\Lambda$$

similarity $\rightarrow X^{-1}AX = \Lambda$

A curved arrow is drawn under the equation $X^{-1}AX = \Lambda$, pointing from the X term to the Λ term, indicating the similarity transformation.

Transformations

Suppose $A\vec{x} = \lambda\vec{x}$ ($\vec{x} \neq \vec{0}$)

$$(A + \sigma I)\vec{x} = (\lambda + \sigma)\vec{x} \quad \text{"shift"}$$

$$A^2\vec{x} = \lambda^2\vec{x}$$

$$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$