February 23, 2026

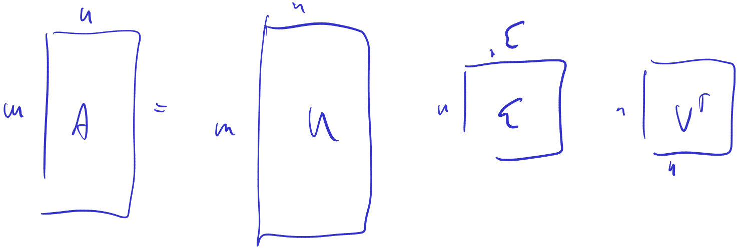
Announcements

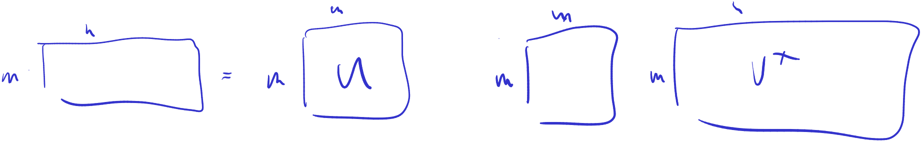
- IEF
- Office hour shuffle
- HW3 typo fix
- Formula sheet for exam 2
- Exam 1 second page grades available

Goals

- LSQ
  - rank-deficient
  - SVD-fast
- Eigenvalues

SVD

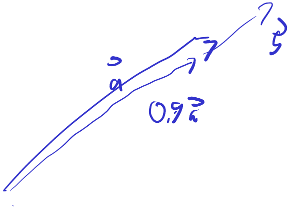




"economy / reduced"

Numerical rank

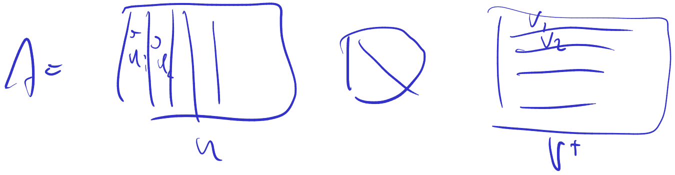
$$\left[\,\vec{a} \,\middle|\, \vec{b}\,\right]$$



rank : not computable

$$\text{num. rank}_\varepsilon(A) = \#\{\sigma_i \geq \varepsilon \;:\; \sigma_i \text{ singular values of } A\}$$

Low rank approx



$$= \sum_{i=1}^{n} \vec{u}_i \vec{v}_i^T \sigma_i$$

Theorem (Eckart-Young-Mirsky)

If $k < r = \text{rank}(A)$ *and*

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T, \quad \textit{then}$$

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

$$\min_{\text{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k+1}^{n} \sigma_j^2}.$$

Rank-deficient LSQ

$$A\vec{x} \cong \vec{b}$$

rank-deficient.

$$\|A\vec{x} - \vec{b}\|_2 = \|U\Sigma V^T\vec{x} - \vec{b}\|_2$$

$$= \|\Sigma \underbrace{V^T\vec{x}}_{\vec{y}} - U^T\vec{b}\|_2$$

$$= \|\Sigma\vec{y} - U^T\vec{b}\|_2$$

$$\left\| \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_{r} & \\ & & & & 0 \end{bmatrix}}_{\Sigma} \tilde{y} - U^T b \right\|_2$$

$$V^T x = \tilde{y} \qquad y_i = \begin{cases} (U^T b)_i / \sigma_i & \text{if } \sigma_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & & & \\ & \ddots & & & \\ & & 1/\sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \qquad \underline{\underline{A^+ = V \Sigma^+ U^T}}$$

Eigenvalue

$$\vec{x}(t)$$

$$\vec{F}(t) = A\,\vec{x}(t)$$

$$F = ma = m\frac{\partial^2 \vec{x}}{\partial t^2}$$

$$\frac{\partial^2 \vec{x}}{\partial t^2} = A\,\vec{x}(t)$$

$$\vec{x}(t) = \vec{x}_0 \sin(\omega t)$$

$$-\omega^2\,\vec{x}_0 \sin\cancel{(\omega t)} = A\,\vec{x}_0 \sin\cancel{(\omega t)}$$

"Spectral radius" = mag. of biggest eigenvalue

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

$$A x_i = \lambda_i x_i \quad (\vec{x}_i \neq \vec{0})$$

$$\downarrow$$

$$AX = X\Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

similarity $\rightarrow$ $X^{-1} A X = \Lambda$

Transformations

Suppose $A\vec{x} = \lambda\vec{x} \quad (\vec{x} \neq \vec{0})$

$$(A + \sigma I)\vec{x} = (\lambda + \sigma)\vec{x} \qquad \text{"shift"}$$

$$A^2\vec{x} = \lambda^2\vec{x}$$

$$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$