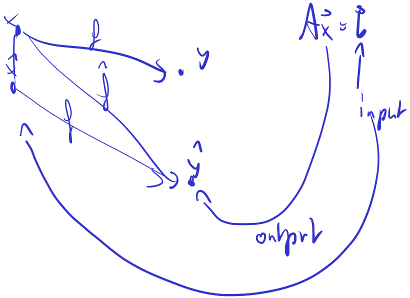
February 2, 2026

Announcements

- Chicago: office hours, exam tomorrow-ish



backward error is perturbed b:

$\vec{r} = \vec{b} - A\vec{x}$

Goals

cond nr upper/lower:

$$\frac{\|\Delta \text{output}\|}{\|\text{output}\|} \le \kappa \frac{\|\Delta \text{input}\|}{\|\text{input}\|}$$

upper bond on perturbation in output.

in a specific scenario have ... have

↳ gives lower bond on κ

"Well conditioned"

Goals:

- matrix norms
- condition of solve and matrices
- residual
- matrix change.

Matrix norms

$A\vec{x}$

$\|A\|$

$$\|A\vec{x}\| \le \underbrace{?}_{\infty?} \|\vec{x}\|$$

$$\|Ax\| \le \|A\| \, \|x\|$$

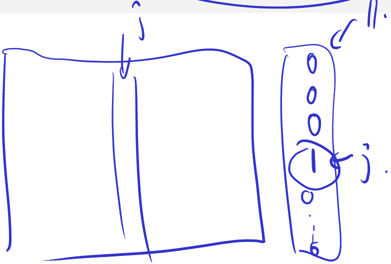
"submultiplicativity"

$$\frac{\|Ax\|}{\|x\|} \le \|A\| \quad (\vec{x} \ne \vec{0})$$

$$\max_{\vec{x} \ne \vec{0}} \text{—"—} =: \|A\|$$

$$\max_{\vec{x} \ne \vec{0}} \left\| A \frac{x}{\|x\|} \right\| = \max_{\|\vec{y}\|=1} \|A\vec{y}\|$$

$$\|A\|_1 = \max_{\text{col } j} \sum_{\text{row } i} |A_{i,j}| ,$$



$\|\cdot\|_1 = 1$

Scenario:

$Ax = b$

$A\Delta x = \Delta b$

$A(x+\Delta x) = b + \Delta b$

$$
\begin{aligned}
\frac{\text{rel err. in output}}{\text{rel err. in input}} &= \frac{\|\Delta \boldsymbol{x}\| \,/\, \|\boldsymbol{x}\|}{\|\Delta \boldsymbol{b}\| \,/\, \|\boldsymbol{b}\|} = \frac{\|\Delta \boldsymbol{x}\|\ \|\boldsymbol{b}\|}{\|\Delta \boldsymbol{b}\|\ \|\boldsymbol{x}\|} \\
&= \frac{\|A^{-1}\Delta \boldsymbol{b}\|\ \|A\boldsymbol{x}\|}{\|\Delta \boldsymbol{b}\|\ \|\boldsymbol{x}\|} \\
&\le \|A^{-1}\|\ \|A\| \cancel{\frac{\|\Delta \boldsymbol{b}\|\ \|\boldsymbol{x}\|}{\|\Delta \boldsymbol{b}\|\ \|\boldsymbol{x}\|}} \\
&= \|A^{-1}\|\ \|A\| .
\end{aligned}
$$

could dep. on b, but didn't.

solve: $A x = b$

$x \uparrow$ out, $b \uparrow$ in

$\operatorname{cond}(\text{solve}) \leq \|A\| \, \|A^{-1}\|$

matvec: $A x = b$

$x \uparrow$ in, $b \uparrow$ out

$\operatorname{cond}(\text{matvec})$

$\Leftrightarrow$ solve* $A^{-1} b = x$

$b \uparrow$ out, $x \uparrow$ in

$\operatorname{cond}(\text{solve}^*) = \|A^{-1}\|$

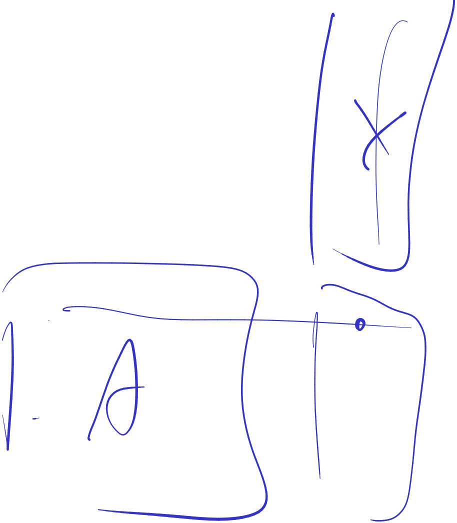
$\|(A^{-1})^{-1}\|$

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\widehat{\boldsymbol{x}}\|} \leq \operatorname{cond}(A) \frac{\|\Delta A\|}{\|A\|}.$$

computationally available!

$$(A + \Delta A)(x + \Delta x) = (b + \Delta b)$$

$\Delta A \; \Delta x$ small?



$$A^{-1} A \vec{x} = x$$

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6a \\ 2b \\ -3c \end{pmatrix}$$

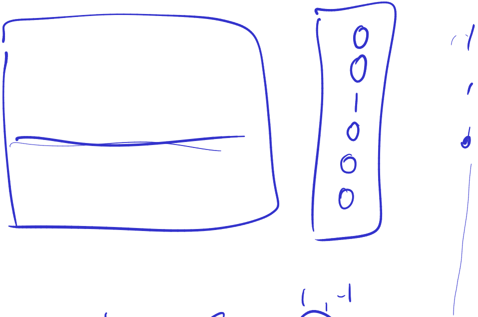
```
rs = np.sum(np.abs(A), axis=1)
i = np.argmax(rs)
```

$$= \max_{col} \sum_{row} |A_{ij}|$$

$$(Ax)_i = \sum_j A_{ij} x_j$$

$$\|x\|_\infty = 1$$

$$\max \cdot 1 \le 1$$



$$(Ax)_i = \sum A_{ij} x_j$$

$$\|Ax\|_\infty = \max_i \left| \quad \right|$$