February 4, 2026

Announcements

$O(n^2)$ $O(n^3)$

Goals

Orthogonality / SVDs

2-norm

Linear system solving

LU

Pivoting.

$$\|\vec{x}\|_2^2 = \vec{x}^T \vec{x}$$

$$\|Ax\|_2^2 = (A\vec{x})^T (A\vec{x}) = \vec{x}^T (A^T A) \vec{x}$$

$\sqrt{\text{eigenvalues of } A^T A}$

→ singular values

diagonal ≥ 0

$$\boxed{A = U \Sigma V^T}$$

orthogonal

$$= \vec{x}^T V \Sigma \cancel{U^T U} \Sigma V^T \vec{x}$$

$$= x^T V \Sigma^2 V^T x$$

Orthogonal matrices U have 2-norm cond nr. 1

$U^T U = I \qquad U U^T = I$

$$\|Qx\|_2^2 = x^T \underline{Q^T Q} x = x^T x = \|x\|_2^2$$

$$\|x^T Q^T\| = \|x\|_2$$

$$\|QA\|_2 = \|A\|_2$$

$$\|AQ\|_2 = \|A\|_2$$

$$\overset{\downarrow}{\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \end{pmatrix}}$$

$$\|A\|_2 = \|\underline{U}\Sigma V^T\|_2 = \|\Sigma \underline{V^T}\|_2 = \|\overset{\downarrow}{\Sigma}\|_2 = \sigma_1$$

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n$$

$$A = U\Sigma V^T \qquad A^{-1} = V\Sigma^{-1}U^T$$

Frobenius

$$\|I\|_F = \sqrt{n}$$

$$\|I\|_{\text{induced}} = 1$$

$$\|A\|_F = \|U\Sigma V^T\|_F = \|\Sigma\|_F = \sqrt{\Sigma \sigma_i^2}$$

Why not GE?

$$A = LU$$

$$Ax = b$$

$$O(n^2) \rightarrow L\underbrace{Ux}_{y} = b$$

$$O(n^2) \rightarrow Ux = y$$

RREF

- rank-revealing
- LU is not

$$PA = LU$$

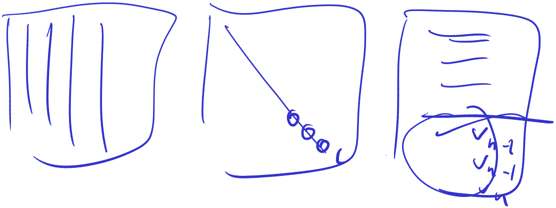
$$A = P^T LU$$

$$\det A = \boxed{\det P^T}\, \cancel{\det L}\, \det U$$

$$N(A) = \{ x ; Ax = 0 \}$$

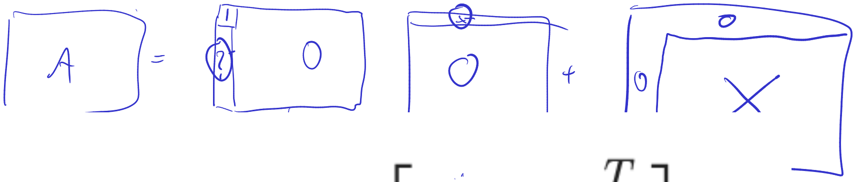
$$A = U \Sigma V^T$$

$=$



$$\vec{x} = \alpha_{n-2} \vec{v}_{n-2} + \dots \alpha_n \vec{v}_n$$

$$V^T \vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{n-2} \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix}$$



$$\begin{bmatrix} u_{00} & \boldsymbol{u}_{01}^T \\ & U_{11} \end{bmatrix}$$

U

$$\begin{bmatrix} 1 & \\ \boldsymbol{l}_{10} & L_{11} \end{bmatrix} \quad \begin{bmatrix} a_{00} & \boldsymbol{a}_{01} \\ \boldsymbol{a}_{10} & A_{11} \end{bmatrix}$$

L

A

L·U = A

for $i = 0 \ldots (\#\text{columns} - 1)$

	Cost	
▶ $u_{11} = a_{11}$, $\boldsymbol{u}_{12}^T = \boldsymbol{a}_{12}^T$.	$n-i$	$O(n)$
▶ $\ell_{21} = \boldsymbol{a}_{21}/u_{11}$.	$n-i-1$	$O(n)$
▶ $L_{22}U_{22} = A_{22} - \ell_{21}\boldsymbol{u}_{12}^T$.	$(n-i-1)^2$	$O(n^2)$
		$O(n^2)$

$n \times$

$O(n^3)$

$$A^{-1} = V \Sigma^{-1} U^T \qquad V\left(\Sigma^{-1}\left(U^T \quad x\right)\right.$$