

February 4, 2026
Announcements

$$O(n^2) \quad O(n^2)$$

Goals

Orthogonality / SVDs
2-norm

Linear system solving
LU
Pivoting.

$$\|x\|_2^2 = x^T x$$

$$\|Ax\|_2^2 = (Ax)^T (Ax) = x^T (A^T A)x$$

diagonal ≥ 0

$$A = U \Sigma V^T$$

orthogonal

Eigenvalues of $A^T A$
 \rightarrow singular values

$$= x^T V \cancel{U^T U} V^T x$$

$$= x^T V \Sigma^2 V^T x$$

2-norm

Orthogonal matrices U have cond n.r. 1

$$U^T U = I \quad U U^T = I$$

$$\|Qx\|_2^2 = x^T \underbrace{Q^T Q}_{I} x = \tilde{x}^T \tilde{x} = \|\tilde{x}\|_2^2$$

$$\|x^T Q^T\| = \|x\|_2$$

$$\|QA\|_2 = \|A\|_2$$

$$\|AQ\|_2 = \|A\|_2$$

$$\|A\|_2 = \|U \Sigma V^T\|_2 = \|\Sigma V^T\|_2 = \|\Sigma\|_2 = \sigma_1$$

\downarrow
 $(\sigma_1, \sigma_2, \sigma_3, \dots)$

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n$$

$$A = U \Sigma V^T \quad A^{-1} = V \Sigma^{-1} U^T$$

Frobenius

$$\|I\|_F = \sqrt{n}$$

$$\|I\|_{\text{induced}} = 1$$

$$\|A\|_F = \|U \Sigma V^T\|_F = \|\Sigma\|_F = \sqrt{\sum \sigma_i^2}$$

Why not GE?

$$A = LU$$

$$Ax = b$$

$$O(n^2) \rightarrow \underbrace{LU}_y x = b$$

$$O(n^2) \rightarrow Ux = y$$

$$PA = LU$$

$$A = P^T LU$$

$$\det A = \det P^T \det L \det U$$

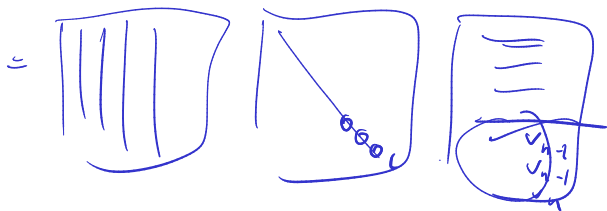
RREF

• rank-revealing

• LU is not

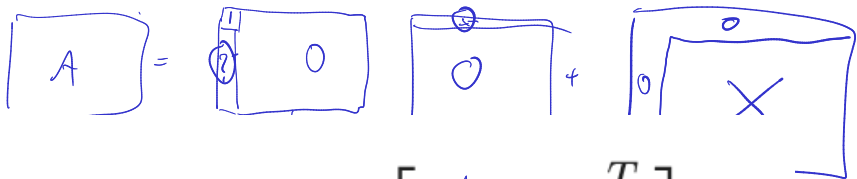
$$N(A) = \{ x; Ax=0 \}$$

$$A = U \Sigma V^T$$



$$\vec{x} = \alpha_{n-1} \vec{v}_{n-1} + \dots + \alpha_n \vec{v}_n$$

$$V^T \vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix}$$



$$\begin{bmatrix} 1 & \\ \mathbf{l}_{10} & \mathbf{L}_{11} \end{bmatrix} \quad \begin{bmatrix} u_{00} & \mathbf{u}_{01}^T \\ \mathbf{a}_{00} & \mathbf{a}_{01} \\ \mathbf{a}_{10} & \mathbf{A}_{11} \end{bmatrix} \quad \begin{bmatrix} U_{11} \end{bmatrix}$$

Diagram illustrating the decomposition of a matrix A into a product of three matrices:

$$\begin{bmatrix} 1 & \\ \mathbf{l}_{10} & \mathbf{L}_{11} \end{bmatrix} \begin{bmatrix} u_{00} & \mathbf{u}_{01}^T \\ \mathbf{a}_{00} & \mathbf{a}_{01} \\ \mathbf{a}_{10} & \mathbf{A}_{11} \end{bmatrix} \begin{bmatrix} U_{11} \end{bmatrix}$$

The diagram shows the relationship between the matrices L , U , and A through their components. A blue line connects the 1 in the top-left of L to the u_{00} in the top-left of U . Another blue line connects the \mathbf{l}_{10} in the bottom-left of L to the \mathbf{a}_{10} in the bottom-left of U . The \mathbf{a}_{00} and \mathbf{a}_{10} elements in U are circled, and the \mathbf{a}_{01} and \mathbf{A}_{11} elements in U are also circled. A blue line connects the \mathbf{a}_{00} and \mathbf{a}_{10} elements to the \mathbf{a}_{01} and \mathbf{A}_{11} elements. The \mathbf{a}_{00} and \mathbf{a}_{10} elements are labeled A , and the \mathbf{a}_{01} and \mathbf{A}_{11} elements are labeled U .

$$L \cdot U = A$$

for $i=0 \dots (\#columns-1)$

► $u_{11} = a_{11}, \mathbf{u}_{12}^T = \mathbf{a}_{12}^T.$

► $\ell_{21} = \mathbf{a}_{21} / u_{11}.$

► $L_{22}U_{22} = A_{22} - \ell_{21}\mathbf{u}_{12}^T.$

Cost	
$n-i$	$O(n)$
$n-i-1$	$O(n)$
$(n-i-1)^2$	$O(n^2)$
	$O(n^2)$

$n \times n$

$O(n^3)$

$$A^{-2} = U \Sigma^{-2} U^T \quad U \Sigma^{-1} U^T x$$