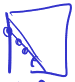
March 2, 2026

Announcements

- HW4
- Exam 2 + second chance

Goals

- all eigenvalues, eigenvectors of a matrix
- simul
- orth
- QR iteration
- 
- shifts in QR iteration

QR factorization

$$A = QR$$

not unique, up to signs

$$\cancel{X_0 = \begin{bmatrix} | & | & | & | & | \end{bmatrix}}$$

$$\cancel{X_1 = A X_0}$$

$$\cancel{X_2 = A X_1}$$

$X_0 = \boxed{\quad}$

$Q_0 R_0 = X_0$

$X_1 = A Q_0 \quad \to \quad Q_1 R_1 Q_0^T = A$

$Q_1 R_1 = X_1$

$X_2 = A Q_1$

$\|Q_{i+1} - Q_i\|_F^2 < \varepsilon$ iffy

$\hat{X}_n := Q_n^T A Q_n$

QR iteration

$$\bar X_0 = A$$

$$\bar Q_k \bar R_k = \bar X_k \iff R_k = \bar Q_k^T \bar X_k$$

$$\bar X_{k+1} = \bar R_k \bar Q_k$$

$$\bar X_0 = A$$

$$\bar X_{k+1} = \bar Q_k^T \bar R_k \bar R_k \bar Q_k \bar X_k \iff \bar X_k \bar Q_k$$

$$\bar X_{k+1} = \bar R_k \bar Q_k$$

$$\hat X_k = \bar X_{k+1}$$

$$Q_k = \bar Q_0 \cdots \bar Q_k$$

$$A^2 = \bar Q_0 \underbrace{\bar R_0 \bar Q_0} \bar R_0 = \bar Q_0 \bar Q_1 \bar R_1 \bar R_0$$

Proof sketch: Equivalence of QR iteration/Orth. iteration

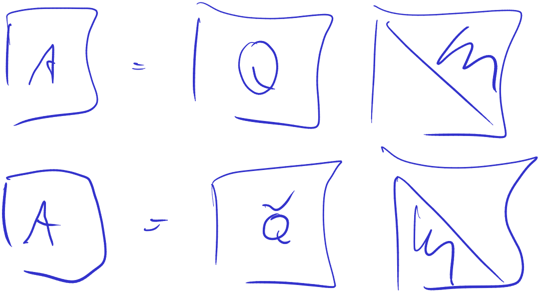
Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$,
    and because of $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{X}_1 = Q_0 \bar{Q}_1 \bar{R}_1$
    we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

QL factorization



$$\bar{X}_0 = A$$

$$\tilde{Q}_k \bar{R}_k = \bar{X}_k$$

$$\bar{X}_{k+1} = \bar{R}_k \tilde{Q}_k$$

$$\bar{X}_0^{-H} = A^{-H}$$

$$\tilde{Q}_k \bar{R}_k^{-H} = \bar{X}_k^{-H}$$

$$\bar{X}_{k+1}^{-H} = \bar{R}_k^{-H} \tilde{Q}_k$$

$$\bar{X}_0 = A$$

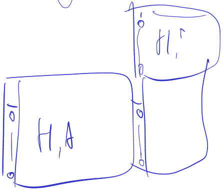
$$\tilde{Q}_k \bar{R}_k = \bar{X}_k - \sigma_k I$$

$$\bar{X}_{k+1} = \bar{R}_k \tilde{Q}_k + \sigma_k I$$

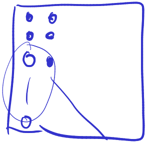
$$\left(\bar{X}_0^{-1}\right)^H = \bar{X}_0^{-H} \quad \left(X_0^H\right)^{-1}$$

$$(H A) H^T$$

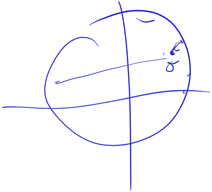
$$H = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \text{mini-Householder} \\ 0 & \end{pmatrix}$$

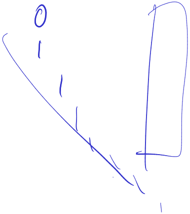


$$H_{n-1} \cdots H_1 A H_1^T \cdots H_{n-1}^T$$



upper Hessenberg Form





$$A\vec{x} = \lambda \vec{x} \quad (x \neq 0)$$

$$= U \Sigma V^T$$

$$A = Q \Lambda Q^T$$

$$= X D X^{-1}$$

$$B = X A X^{-1}$$

$$y = X \vec{x}$$

$$B\vec{y} = X A \cancel{X^{-1} X} \vec{x} = X A \vec{x} = X(\lambda x)$$

$$= \lambda \vec{y}$$