March 4, 2026

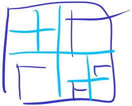
Announcements

- exam drop
- HW 4
- QR shift

Goals

- Krylov
  - ↳ eigenvalues
    Arnoldi
- SVD
- Nonlinear

Krylov

- matrix stored
  (funny format)
  ↳ sparse
  ↳ 

- no matrix

```
f(x)
    return A@x
```

$\vec{x}_0$

$\vec{x}_1 = A\vec{x}_0$

$\vdots$

$$V_k = \begin{bmatrix} \vec{x}_0 & \cdots & \vec{x}_{k-1} \end{bmatrix}$$

can fail, by not being lin. indep.

Let $K = K_n$

$$A \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

$$\iff AK = K \begin{bmatrix} 0 & 0 & & & \vdots \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \vdots \end{bmatrix}$$

upper Hessenberg

$$K^{-1} A K = C$$

$K = QR$

columns of this: ONB span Krylov space

$q_0 \cdots q_{n-1}$

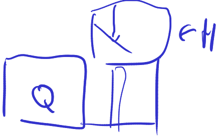
$Q = KR^{-1}$

$Q^T A Q = R \underbrace{K^{-1} A K} R^{-1}$

$= R C R^{-1}$

$=$ upper Hessenberg

$= H$



We find that $Q_n^T A Q_n$ is also upper Hessenberg: $Q_n^T A_n Q_n = H$.
Also readable as $AQ_n = Q_n H$, which, read column-by-column, is:

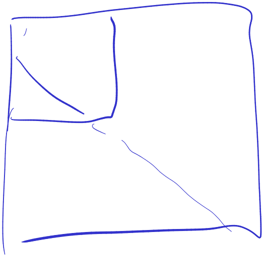
$$A\boldsymbol{q}_k = h_{1k}\boldsymbol{q}_1 + \cdots + h_{k+1,k}\boldsymbol{q}_{k+1}$$

We find: $h_{jk} = \boldsymbol{q}_j^T A \boldsymbol{q}_k$. Use that to rewrite, letting $\boldsymbol{v} = A\boldsymbol{q}_k$:

$$\boldsymbol{v} - (\boldsymbol{q}_1^T \boldsymbol{v})\boldsymbol{q}_1 - \cdots - (\boldsymbol{q}_k^T \boldsymbol{v})\boldsymbol{q}_k = h_{k+1,k}\boldsymbol{q}_{k+1}$$

- Looks just like Gram-Schmidt QR!
- Can compute $(k+1)$st column of H and $\boldsymbol{q}_{k+1}$ from $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k$.

This is called Arnoldi iteration. For symmetric: Lanczos iteration.



"Nik vdwei"

Computing an SVD:

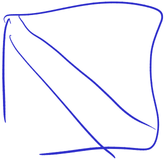
$A = U \Sigma V^T$

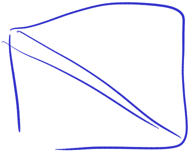
$A^T A = V \Sigma \cancel{U^T U} \Sigma V^T$

$= V \Sigma^2 V^T$

bad

$A = U \Sigma V^T$

$HA =$ 

$H A \tilde{H}^T =$ 

Nonlinear

$$f(\vec{x}) = g(\vec{x})$$

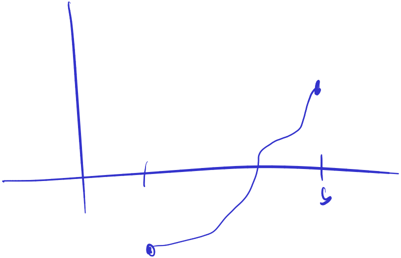
n unknowns

n equations

$$\Leftrightarrow \quad f(\vec{x}) - g(\vec{x}) = \vec{0}$$

$$\Leftrightarrow \quad \tilde{f}(\vec{x}) = \vec{0}$$

"root finding"



"inverse function theorem"

$f : C^1(\mathbb{R}^n)$

$: \mathbb{R}^n \to \mathbb{R}^n$

$J_f(a)$ is invertible

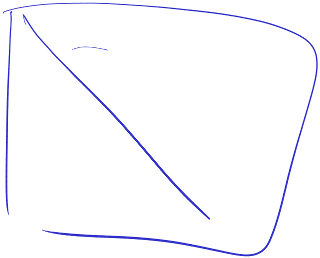
- contraction mappings

$$g : \mathbb{R}^n \to \mathbb{R}^n$$

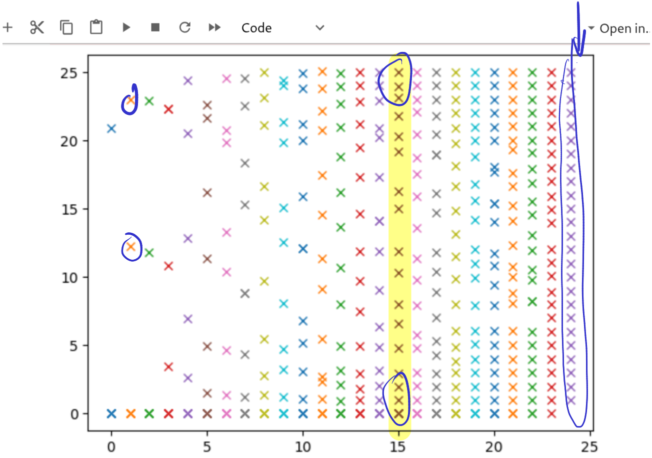
$$\|g(x) - g(y)\| \le \gamma \|x - y\|$$

$0 \le \gamma < 1$

$$g(x^*) = x^*$$



$\vec{e}_1$


Code
Open in..
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