March 9, 2026

Announcements

- HW4 out
- 5 out Thu
- Exam 2 second

Goals

- $f(x)=0$
- rates of convergence
- stopping criteria
- Taylor
- Fixed point
- Newton

Rates of convergence

$$\vec{x}_k \to \vec{x}^*$$

$$\vec{e}_k := \vec{x}_k - x^*$$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \quad \begin{cases} >0 \\ <\infty \end{cases}$$

only relevant in the limit

does not care about any finite number of iterations

$$\|e_{k+1}\| \leq C \|e_k\|^r$$

For bisection: $C = \frac{1}{2}$ $\quad r = 1$

Reliability

$$\|e_{k+1}\| \le \frac{1}{2} \|e_k\|$$

$$\|e_{k+1}\| \le C \cdot \|e_k\|^2$$

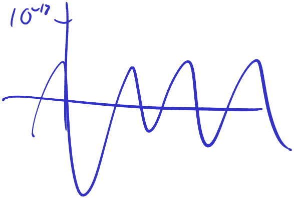
$$2^x = \frac{1}{10}$$

$$x \cdot \log 2 = \log \frac{1}{10}$$

$$e_{k+1} \overset{!}{=} C e_k^r \iff \log e_{k+1} = \log C + r \cdot \log e_k$$

Stopping criteria

- $\|f(x)\| < \epsilon$
- $\|\vec{x}_n - \vec{x}_{n-1}\| < \epsilon$
- $\|\vec{x}_n - \vec{x}_{n-1}\| / \|x_{n-1}\| < \epsilon$



Taylor

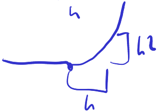
$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(x)}{2!}h^2 + \dots$$

f analytic

$f \in C^0$:

$f \in C^1$:

$f \in C^2$



$f \in C^{k+1}$

$\theta \in (x, x+h)$

$$f(x+h) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}\cdot h^i + \frac{f^{(k+1)}(\theta)}{(k+1)!} h^{k+1}$$

Fixed point (1D)

goal

$$g(x^*) = x^*$$

$$x_0 = ?$$

$$x_{k+1} = g(x_k)$$

$$e_k = x_k - x^*$$

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

$$= g'(x^*) \cdot (x_k - x^*) + g''(x^*) \cdot e_k^2 + \dots$$

$$= g'(x^*) e_k + g''(x^*) e_k^2 + \dots$$

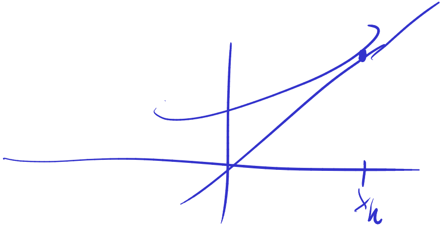
If $g'(x^*) < 1$: convergence

$g'(x^*) = 0$: quadratic convergence

Newton

$x_0 = ?$

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)}$$



$$f(x_n+h) \approx f(x_n) + h \cdot f'(x_n)$$

$$f(x_n) + h \cdot f'(x_n) \overset{!}{=} 0$$

$$h = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)}$$

$$g'(x) = \cancel{1} - \frac{\cancel{f'f'} - f f''}{(f')^2} = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$$g'(x^*) = \frac{0\, f''}{f'} = 0$$