Parallel Numerical Algorithms

Chapter 7 – Differential Equations
Section 7.1 – Ordinary Differential Equations

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CS 554 / CSE 512

Outline

1. Ordinary Differential Equations
   - Parallelism in Solving ODEs
   - Waveform Relaxation
   - Boundary Value Problems for ODEs

Ordinary Differential Equations

Minor potential sources of parallelism in solving initial value problem for system of ODEs $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ include

- For multi-stage methods (e.g., Runge-Kutta), computation of multiple stages in parallel
- For multi-level methods (e.g., extrapolation), computation of multiple levels (e.g., with different step sizes) in parallel
- For multi-rate methods, integration of slowly and rapidly varying components of solution in parallel

Ordinary Differential Equations

Major potential sources of parallelism in solving initial value problem for system of ODEs $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ include

- Evaluation of right-hand-side function $\boldsymbol{f}$ in parallel (e.g., evaluation of forces for n-body problems)
- Parallel implementation of linear algebra computations (e.g., solving linear system in Newton's method for stiff ODEs)
- Partitioning equations in system of ODEs into multiple tasks (e.g., waveform relaxation, discussed next)

Picard Iteration

- Consider initial value problem for system of n ODEs $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$, $t \geq t_0$, with IC $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$
- Starting with $\boldsymbol{y}_0(t) \equiv \boldsymbol{y}_0$, *Picard iteration* is given by

$$\boldsymbol{y}_{k+1}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{y}_k(s))\, ds$$

- If $\boldsymbol{f}$ satisfies Lipschitz condition, then Picard iteration converges to solution of IVP
- Convergence may be slow, but parallelism is excellent, as problem decouples into n independent 1-D quadratures

Waveform Relaxation

- Picard iteration is simple fixed-point iteration on function space
- Picard iteration is often too slow to be useful, but other such iterations may be more rapidly convergent
- Iterative methods of this type are commonly called *waveform relaxation*

Jacobi Waveform Relaxation

- For $n = 2$, consider iteration

$$\begin{bmatrix} y_1^{(k+1)}(t) \\ y_2^{(k+1)}(t) \end{bmatrix}' = \begin{bmatrix} f_1(t, y_1^{(k+1)}(t), y_2^{(k)}(t)) \\ f_2(t, y_1^{(k)}(t), y_2^{(k+1)}(t)) \end{bmatrix}$$

- System of two independent ODEs can be solved in parallel
- Method generalizes in obvious way to arbitrary system of n ODEs and decouples system into n independent ODEs
- Because of its analogy to Jacobi iteration for linear algebraic systems, method is called *Jacobi waveform relaxation*

Gauss-Seidel Waveform Relaxation

- Convergence rate of Jacobi waveform relaxation is improved by *Gauss-Seidel waveform relaxation*, illustrated here for $n = 2$

$$\begin{bmatrix} y_1^{(k+1)}(t) \\ y_2^{(k+1)}(t) \end{bmatrix}' = \begin{bmatrix} f_1(t, y_1^{(k+1)}(t), y_2^{(k)}(t)) \\ f_2(t, y_1^{(k+1)}(t), y_2^{(k+1)}(t)) \end{bmatrix}$$

- Unfortunately, system is no longer decoupled, so parallelism is lost unless components are reordered, analogous to red-black or multicolor ordering

- More generally, multi-splittings can further enhance parallelism in waveform relaxation methods

Boundary Value Problems for ODEs

Potential sources of parallelism in solving boundary value problems for ODEs include

- For finite difference and finite element methods, parallel implementation of resulting linear algebra computations (e.g., cyclic reduction for tridiagonal systems)
- Multi-level methods
- Multiple shooting method

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