CS 598 EVS: Tensor Computations

Tensor Computations

Edgar Solomonik

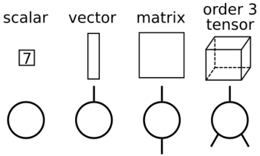
University of Illinois at Urbana-Champaign

Tensors

Tensors

A *tensor* is a collection of elements

- *its dimensions define the size of the collection*
- *its order is the number of different dimensions*
- *specifying an index along each tensor mode defines an element of the tensor*



A few examples of tensors are

- *Order 0 tensors are scalars, e.g.,* $s \in \mathbb{R}$
- *Order 1 tensors are vectors, e.g.,* $\boldsymbol{v} \in \mathbb{R}^n$
- *Order 2 tensors are matrices, e.g.,* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
- *An order 3 tensor with dimensions $s_1 \times s_2 \times s_3$ is denoted as $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{s_1 \times s_2 \times s_3}$ with elements t_{ijk} for $i \in \{1, \ldots, s_1\}, j \in \{1, \ldots, s_2\}, k \in \{1, \ldots, s_3\}$*

Applications of Tensors

Tensors provide a mathematical formalism for multidimensional data

- tensors arise naturally from discretization of equations with multiple variables
- data that can be tabulated according to multiple parameters is representible by a tensor
- numerical simulations with regular grids represent the solution as tensors (typically order 3) and apply discretized operators, which are structured tensors (of order 6 if the grid is 3D)
- higher-order tensors may arise from higher-order (many-body) interactions, such as in quantum chemistry
- tensor decompositions provide general techniques for approximations and analysis of such tensors

Reshaping Tensors

Its often helpful to use alternative views of the same collection of elements

- *Folding a tensor yields a higher-order tensor with the same elements*
- *Unfolding a tensor yields a lower-order tensor with the same elements*
- *In linear algebra, we have the unfolding* $\boldsymbol{v} = \mathit{vec}(\boldsymbol{A})$, *which stacks the columns of* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *to produce* $\boldsymbol{v} \in \mathbb{R}^{mn}$
- *For a tensor* $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{s_1\times s_2\times s_3}$, $\boldsymbol{v} = \mathit{vec}(\boldsymbol{\mathcal{T}})$ *gives* $\boldsymbol{v} \in \mathbb{R}^{s_1 s_2 s_3}$ *with*

$$v_{i+(j-1)s_1+(k-1)s_1 s_2} = t_{ijk}$$

- *A common set of unfoldings is given by matricizations of a tensor, e.g., for order 3,*

$$\boldsymbol{T}_{(1)} \in \mathbb{R}^{s_1\times s_2 s_3}, \boldsymbol{T}_{(2)} \in \mathbb{R}^{s_2\times s_1 s_3}, \quad \textit{and} \quad \boldsymbol{T}_{(3)} \in \mathbb{R}^{s_3\times s_1 s_2}$$

Tensor Transposition

For tensors of order $\geqslant 3$, there is more than one way to transpose modes

- *A* *tensor transposition* *is defined by a permutation* $\boldsymbol{p}$ *containing elements* $\{1, \ldots, d\}$

$$y_{i_{p_1}, \ldots, i_{p_d}} = x_{i_1, \ldots, i_d}$$

- *In this notation, a transposition* $\boldsymbol{A}^T$ *of matrix* $\boldsymbol{A}$ *is defined by* $\boldsymbol{p} = [2, 1]$ *so that*

$$b_{i_2 i_1} = a_{i_1 i_2}$$

- *Tensor transposition is a convenient primitive for manipulating multidimensional arrays and mapping tensor computations to linear algebra*
- *When elementwise expressions are used in tensor algebra, indices are often carried through to avoid transpositions*

Tensor Symmetry

We say a tensor is *symmetric* if $\forall j, k \in \{1, \dots, d\}$

$$t_{i_1 \dots i_j \dots i_k \dots i_d} = t_{i_1 \dots i_k \dots i_j \dots i_d}$$

A tensor is *antisymmetric* (skew-symmetric) if $\forall j, k \in \{1, \dots, d\}$

$$t_{i_1 \dots i_j \dots i_k \dots i_d} = (-1) t_{i_1 \dots i_k \dots i_j \dots i_d}$$

A tensor is *partially-symmetric* if such index interchanges are restricted to be within disjoint subsets of $\{1, \dots, d\}$, e.g., if the subsets for $d = 4$ and $\{1, 2\}$ and $\{3, 4\}$, then

$$t_{ijkl} = t_{jikl} = t_{jilk} = t_{ijlk}$$

Tensor Sparsity

We say a tensor $\mathcal{T}$ is *diagonal* if for some $\boldsymbol{v}$,

$$t_{i_1,\ldots,i_d} = \begin{cases} v_{i_1} & : i_1 = \cdots = i_d \\ 0 & : \textit{otherwise} \end{cases} = v_{i_1}\delta_{i_1 i_2}\delta_{i_2 i_3}\cdots\delta_{i_{d-1} i_d}$$

- *In the literature, such tensors are sometimes also referred to as ‘superdiagonal’*
- *Generalizes diagonal matrix*
- *A diagonal tensor is symmetric (and not antisymmetric)*

If most of the tensor entries are zeros, the tensor is *sparse*

- *Generalizes notion of sparse matrices*
- *Sparsity enables computational and memory savings*
- *We will consider data structures and algorithms for sparse tensor operations later in the course*

Tensor Products and Kronecker Products

Tensor products can be defined with respect to maps $f : V_f \to W_f$ and $g : V_g \to W_g$

$$h = f \times g \quad \Rightarrow \quad g : (V_f \times V_g) \to (W_f \times W_g), \quad h(x,y) = f(x)g(y)$$

Tensors can be used to represent multilinear maps and have a corresponding definition for a tensor product

$$\boldsymbol{T} = \boldsymbol{X} \times \boldsymbol{Y} \quad \Rightarrow \quad t_{i_1,\ldots,i_m,j_1,\ldots,j_n} = x_{i_1,\ldots,i_m} y_{j_1,\ldots,j_n}$$

The *Kronecker product* between two matrices $\boldsymbol{A} \in \mathbb{R}^{m_1 \times m_2}$, $\boldsymbol{B} \in \mathbb{R}^{n_1 \times n_2}$

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B} \quad \Rightarrow \quad c_{i_2+(i_1-1)n_1, j_2+(j_1-1)n_2} = a_{i_1 j_1} b_{i_2 j_2}$$

corresponds to transposing and unfolding the tensor product

General Tensor Contractions

Given tensor $\mathcal{U}$ of order $s+v$ and $\mathcal{V}$ of order $v+t$, a tensor contraction summing over v modes can be written as

$$w_{i_1\ldots i_s j_1\ldots j_t} = \sum_{k_1\ldots k_v} u_{i_1\ldots i_s k_1\ldots k_v} v_{k_1\ldots k_v j_1\ldots j_t}$$

- *This form omits 'Hadamard indices', i.e., indices that appear in both inputs and the output (as with pointwise product, Hadamard product, and batched mat–mul.)*
- *Other contractions can be mapped to this form after transposition*

Unfolding the tensors reduces the tensor contraction to matrix multiplication

- *Combine (unfold) consecutive indices in appropriate groups of size* s ,t, *or* v
- *If all tensor modes are of dimension* n, *obtain matrix–matrix product* $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ *where* $\boldsymbol{C} \in \mathbb{R}^{n^s \times n^t}$, $\boldsymbol{A} \in \mathbb{R}^{n^s \times n^v}$, *and* $\boldsymbol{B} \in \mathbb{R}^{n^v \times n^t}$
- *Assuming classical matrix multiplication, contraction requires* n^{s+t+v} *elementwise products and* $n^{s+t+v} - n^{s+t}$ *additions*

Properties of Einsums

Given an elementwise expression containing a product of tensors, the operands commute

- *For example* $\boldsymbol{AB} \neq \boldsymbol{BA}$*, but*

$$\sum_k a_{ik} b_{kj} = \sum_k b_{kj} a_{ik}$$

- *Similarly with multiple terms, we can bring summations out and reorder as needed, e.g., for* $\boldsymbol{ABC}$

$$\sum_k a_{ik} (\sum_l b_{kl} c_{lj}) = \sum_{kl} c_{lj} b_{kl} a_{ik}$$

A contraction can be succinctly described by a *tensor diagram*

- *Indices in contractions are only meaningful in so far as they are matched up*
- *A tensor diagram is defined by a graph with a vertex for each tensor and an edge/leg for each index/mode*
- *Indices that are not-summed are drawn by pointing the legs/edges into whitespace*

Matrix-style Notation for Tensor Contractions

The *tensor times matrix* contraction along the mth mode of $\boldsymbol{\mathcal{U}}$ to produce $\boldsymbol{\mathcal{V}}$ is expressed as follows

$$\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{U}} \times_m \boldsymbol{V} \Rightarrow \boldsymbol{W}_{(m)} = \boldsymbol{V}\boldsymbol{U}_{(m)}$$

- *$\boldsymbol{W}_{(m)}$ and $\boldsymbol{U}_{(m)}$ are unfoldings where the mth mode is mapped to be an index into rows of the matrix*
- *To perform multiple tensor times matrix products, can write, e.g.,*

$$\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{U}} \times_1 \boldsymbol{X} \times_2 \boldsymbol{Y} \times_3 \boldsymbol{Z} \Rightarrow w_{ijk} = \sum_{pqr} u_{pqr} x_{ip} y_{jq} z_{kr}$$

The *Khatri-Rao product* of two matrices $\boldsymbol{U} \in \mathbb{R}^{m \times k}$ and $\boldsymbol{V} \in \mathbb{R}^{n \times k}$ products $\boldsymbol{W} \in \mathbb{R}^{mn \times k}$ so that

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{u}_1 \otimes \boldsymbol{v}_1 & \cdots & \boldsymbol{u}_k \otimes \boldsymbol{v}_k \end{bmatrix}$$

The Khatri-Rao product computes the einsum $\hat{w}_{ijk} = u_{ik} v_{jk}$ then unfolds $\hat{\boldsymbol{\mathcal{W}}}$ so that $w_{i+(j-1)n,k} = \hat{w}_{ijk}$

Tensor Contractions

A *tensor contraction* multiplies elements of two tensors and computes partial sums to produce a third, in a fashion expressible by pairing up modes of different tensors, defining *einsum* (term stems from Einstein's summation convention)

tensor contraction	*einsum*	*diagram*
inner product	$w = \sum_i u_i v_i$	
outer product	$w_{ij} = u_i v_{ij}$	
pointwise product	$w_i = u_i v_i$	
Hadamard product	$w_{ij} = u_{ij} v_{ij}$	
matrix multiplication	$w_{ij} = \sum_k u_{ik} v_{kj}$	
batched mat.-mul.	$w_{ijl} = \sum_k u_{ikl} v_{kjl}$	
tensor times matrix	$w_{ilk} = \sum_j u_{ijk} v_{lj}$	

The terms 'contraction' and 'einsum' are also often used when more than two operands are involved

Identities with Kronecker and Khatri-Rao Products

- Matrix multiplication is distributive over the Kronecker product

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{A}\boldsymbol{C} \otimes \boldsymbol{B}\boldsymbol{D}$$

we can derive this from the einsum expression

$$\sum_{kl} a_{ik} b_{jl} c_{kp} d_{lq} = \Big(\sum_{k} a_{ik} c_{kp}\Big)\Big(\sum_{l} b_{jl} d_{lq}\Big)$$

- For the Khatri-Rao product a similar distributive identity is

$$(\boldsymbol{A} \odot \boldsymbol{B})^T(\boldsymbol{C} \odot \boldsymbol{D}) = \boldsymbol{A}^T\boldsymbol{C} * \boldsymbol{B}^T\boldsymbol{D}$$

where $*$ *denotes that Hadamard product, which holds since*

$$\sum_{kl} a_{ki} b_{li} c_{kj} d_{lj} = \Big(\sum_{k} a_{ki} c_{kj}\Big)\Big(\sum_{l} b_{li} d_{lj}\Big)$$

CP Decomposition

- The *canonical polyadic or CANDECOMP/PARAFAC (CP) decomposition* expresses an order d tensor in terms of d factor matrices
  - *For a tensor $\mathcal{T} \in \mathbb{R}^{n \times n \times n}$, the CP decomposition is defined by matrices $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ such that*
  $$t_{ijk} = \sum_{r=1}^{R} u_{ir} v_{jr} w_{kr}$$
  the columns of $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ are generally not orthonormal, but may be normalized, so that
  $$t_{ijk} = \sum_{r=1}^{R} \sigma_r u_{ir} v_{jr} w_{kr}$$
  where each $\sigma_r \geqslant 0$ and $\|\boldsymbol{u}_r\|_2 = \|\boldsymbol{v}_r\|_2 = \|\boldsymbol{w}_r\|_2 = 1$
  - *For an order N tensor, the decomposition generalizes as follows,*
  $$t_{i_1 \dots i_d} = \sum_{r=1}^{R} \prod_{j=1}^{d} u_{i_j r}^{(j)}$$
  - *Its rank is generally bounded by $R \leqslant n^{d-1}$*

CP Decomposition Basics

- The CP decomposition is useful in a variety of contexts
  - *If an exact decomposition with* $R \ll n^{d-1}$ *is expected to exist*
  - *If an approximate decomposition with* $R \ll n^{d-1}$ *is expected to exist*
  - *If the factor matrices from an approximate decomposition with* $R = O(1)$ *are expected to contain information about the tensor data*
  - *CP a widely used tool, appearing in many domains of science and data analysis*
- Basic properties and methods
  - *Uniqueness (modulo normalization) is dependent on rank*
  - *Finding the CP rank of a tensor or computing the CP decomposition is NP-hard (even with* $R = 1$*)*
  - *Typical rank of tensors (likely rank of a random tensor) is generally less than the maximal possible rank*
  - *CP approximation as a nonlinear least squares (NLS) problem and NLS methods can be applied in a black-box fashion, but structure of decomposition motivates alternating least-squares (ALS) optimization*

Alternating Least Squares Algorithm

- The standard approach for finding an approximate or exact CP decomposition of a tensor is the *alternating least squares (ALS) algorithm*
  - *Consider rank* R *decomposition of a tensor* $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$ *over* $\mathbb{R}$
  - *A sweep takes as input* $[\![\boldsymbol{U}^{(k)}, \boldsymbol{V}^{(k)}, \boldsymbol{W}^{(k)}]\!]$ *solves 3 quadratic optimization problems to obtain* $[\![\boldsymbol{U}^{(k+1)}, \boldsymbol{V}^{(k+1)}, \boldsymbol{W}^{(k+1)}]\!]$*, updating each factor matrix in sequence, typically via the normal equations:*

$$\begin{aligned}
(\boldsymbol{V}^{(k)T}\boldsymbol{V}^{(k)} * \boldsymbol{W}^{(k)T}\boldsymbol{W}^{(k)})\boldsymbol{U}^{(k+1)} &= \boldsymbol{T}_{(1)}(\boldsymbol{V}^{(k)} \odot \boldsymbol{W}^{(k)}) \\
(\boldsymbol{U}^{(k+1)T}\boldsymbol{U}^{(k+1)} * \boldsymbol{W}^{(k)T}\boldsymbol{W}^{(k)})\boldsymbol{V}^{(k+1)} &= \boldsymbol{T}_{(2)}(\boldsymbol{U}^{(k+1)} \odot \boldsymbol{W}^{(k)}) \\
(\boldsymbol{U}^{(k+1)T}\boldsymbol{U}^{(k+1)} * \boldsymbol{V}^{(k+1)T}\boldsymbol{V}^{(k+1)})\boldsymbol{W}^{(k+1)} &= \boldsymbol{T}_{(3)}(\boldsymbol{U}^{(k+1)} \odot \boldsymbol{V}^{(k+1)})
\end{aligned}$$

  - *Residual decreases monotonically, since the subproblems in each subset of* nR *variables are quadratic*
  - *Forming the linear equations has cost* $O(dnR^2)$ *while forming the right-hand-sides requires an MTTKRP with cost* $O(n^dR)$

Tucker Decomposition

- The *Tucker decomposition* expresses an order d tensor via a smaller order d core tensor and d factor matrices
  - *For a tensor $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{n\times n\times n}$, the Tucker decomposition is defined by core tensor $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{R_1\times R_2\times R_3}$ and factor matrices $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ with orthonormal columns, such that*

$$t_{ijk} = \sum_{p=1}^{R_1}\sum_{q=1}^{R_2}\sum_{r=1}^{R_3} z_{pqr}u_{ip}v_{jq}w_{kr}$$

  - *For general tensor order, the Tucker decomposition is defined as*

$$t_{i_1\dots i_d} = \sum_{r_1=1}^{R_1}\cdots\sum_{r_d=1}^{R_d} z_{r_1\dots r_d}\prod_{j=1}^{d} u^{(j)}_{i_j r_j}$$

    which can also be expressed as

$$\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{Z}} \times_1 \boldsymbol{U}^{(1)} \cdots \times_d \boldsymbol{U}^{(d)}$$

  - *The Tucker ranks, (R_1, R_2, R_3) are each bounded by the respective tensor dimensions, in this case, $R_1, R_2, R_3 \leqslant n$*
  - *In relation to CP, Tucker is formed by taking all combinations of tensor products between columns of factor matrices, while CP takes only disjoint products*

Tucker Decomposition Basics

- The Tucker decomposition is used in many of the same contexts as CP
  - *If an exact decomposition with each $R_j < n$ is expected to exist*
  - *If an approximate decomposition with $R_j < n$ is expected to exist*
  - *If the factor matrices from an approximate decomposition with $R = O(1)$ are expected to contain information about the tensor data*
  - *Tucker is most often used for data compression and appears less often than CP in theoretical analysis*
- Basic properties and methods
  - *The Tucker decomposition is not unique (can pass transformations between core tensor and factor matrices, which also permit their orthogonalization)*
  - *Finding the best Tucker approximation is NP-hard (for $R = 1$, CP = Tucker)*
  - *If an exact decomposition exists, it can be computed by high-order SVD (HOSVD), which performs d SVDs on unfoldings*
  - *HOSVD obtains a good approximation with cost $O(n^{d+1})$ (reducible to $O(n^d R)$ via randomized SVD or QR with column pivoting)*
  - *Accuracy can be improved by iterative nonlinear optimization methods, such as high-order orthogonal iteration (HOOI)*

Tensor Train Decomposition

- The *tensor train decomposition* expresses an order d tensor as a chain of products of order 2 or order 3 tensors
  - *For an order 4 tensor, we can express the tensor train decomposition as*

$$t_{ijkl} = \sum_{p,q,r} u_{ip} v_{pjq} w_{qkr} z_{rl}$$

  - *More generally, the Tucker decomposition is defined as follows,*

$$t_{i_1 \dots i_d} = \sum_{r_1=1}^{R_1} \cdots \sum_{r_{d-1}=1}^{R_{d-1}} u^{(1)}_{i_1 r_1} \left(\prod_{j=2}^{d-1} u^{(j)}_{r_{j-1} i_j r_j} \right) u^{(d)}_{r_{d-1} i_d}$$

  - *In physics literature, it is known as a matrix product state (MPS), as we can write it in the form,*

$$t_{i_1 \dots i_d} = \langle \boldsymbol{u}^{(1)}_{i_1}, \boldsymbol{U}^{(2)}_{i_2} \cdots \boldsymbol{U}^{(d-1)}_{i_{d-1}} \boldsymbol{u}^{(d)}_{i_d} \rangle$$

  - *For an equidimensional tensor, the ranks are bounded as $R_j \leqslant \min(n^j, n^{d-j})$*

Tensor Train Decomposition Basics

- Tensor train has applications in quantum simulation and in numerical PDEs
  - *Its useful whenever the tensor is low-rank or approximately low-rank, i.e., $R_j R_{j+1} < n^{d-1}$ for all $j < d-1$*
  - *MPS (tensor train) and extensions are widely used to approximate quantum systems with $\Theta(d)$ particles/spins*
  - *Often the MPS is optimized relative to an implicit operator (often of a similar form, referred to as the matrix product operator (MPO))*
  - *Operators and solutions to some standard numerical PDEs admit tensor-train approximations that yield exponential compression*
- Basic properties and methods
  - *The tensor train decomposition is not unique (can pass transformations, permitting orthogonalization into canonical forms)*
  - *Approximation with tensor train is NP hard (for $R = 1$, CP = Tucker = TT)*
  - *If an exact decomposition exists, it can be computed by tensor train SVD (TTSVD), which performs $d-1$ SVDs*
  - *TTSVD can be done with the cost $O(n^{d+1})$ or $O(n^d R)$ with faster low-rank SVD*
  - *Iterative (alternating) optimization is generally used when optimizing tensor train relative to an implicit operator or to refine TTSVD*

Summary of Tensor Decomposition Basics

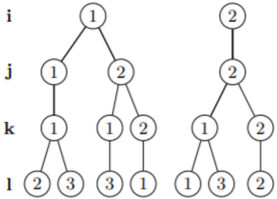
We can compare the aforementioned decomposition for an order d tensor with all dimensions equal to n and all decomposition ranks equal to R

decomposition	CP	Tucker	tensor train
size	dnR	$dnR + R^d$	$2nR + (d-2)nR^2$
uniqueness	if $R \leqslant (3n-2)/2$	no	no
orthogonalizability	none	partial	partial
exact decomposition	NP hard	$O(n^{d+1})$	$O(n^{d+1})$
approximation	NP hard	NP hard	NP hard
typical method	ALS	HOSVD	TT-ALS (implicit)

Sparse Tensor Formats

- The overhead of transposition, and non-standard nature of the arising sparse matrix products, motivates sparse data structures for tensors that are suitable for tensor contractions of interest
  - *Particularly important, especially for tensor decomposition, are MTTKRP (suffices to CP ALS) and TTMc (suffices for HOOI)*
  - *TTM is also prevalent, but is a less attractive primitive in the sparse case than MTTKRP and TTMc, as these yield dense, low-order outputs, while the output of TTM can be sparse and larger than the starting tensor*
- The *compressed sparse fiber (CSF)* format provides an effective representation for sparse tensors
  - *CSF can be visualized as a tree (diagram taken from original CSF paper, by Shaden Smith and George Karpis, IAˆ3, 2015)*

i	j	k	l
1	1	1	2
1	1	1	3
1	2	1	3
1	2	2	1
2	2	1	1
2	2	1	3
2	2	2	2



Operations in Compressed Format

- CSF permits efficient execution of important sparse tensor kernels
  - Analogous to CSR format, which enables efficient implementation of the sparse matrix vector product
  - where `row[i]` stores a list of column indices and nonzeros in the ith row of $\boldsymbol{A}$

```
for i in range(n):
  for (a_ij,j) in row[i]:
    y[i] += a_ij * x[j]
```

  - In CSF format, a multilinear function evaluation $\boldsymbol{f}^{(\boldsymbol{\mathcal{T}})}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{T}_{(1)}(\boldsymbol{x} \odot \boldsymbol{y})$ can be implemented as

```
for (i,T_i) in T_CSF:
  for (j,T_ij) in T_i:
    for (k,t_ijk) in T_ij:
      z[i] += t_ijk * x[j] * y[k]
```

MTTKRP in Compressed Format

- MTTKRP and CSF pose additional implementation opportunities and challenges
  - MTTKRP $u_{ir} = \sum_{j,k} t_{ijk} v_{jr} w_{kr}$ can be implemented by adding a loop over r to our code for $\boldsymbol{f}^{(\mathcal{T})}$, but would then require $3mr$ operations if m is the number of nonzeros in $\mathcal{T}$, can reduce to $2mr$ by amortization

```
for (i,T_i) in T_CSF:
  for (j,T_ij) in T_i:
    for r in range(R):
      f_ij = 0
      for (k,t_ijk) in T_ij:
        f_ij += t_ijk * w[k,r]
      u[i,r] = f_ij * v[j,r]
```

  - However, this amortization is harder (requires storage or iteration overheads) if the index i is a leaf node in the CSF tree
  - Similar challenges in achieving good reuse and obtaining good arithmetic intensity arise in implementation of other kernels, such as TTMc