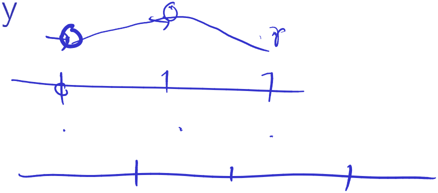
Today

- differencing
- finite diff for hyperbolic

Announcements

HW1 due Friday

Finite Differences Numerically



derivative matrices are "shiftable"

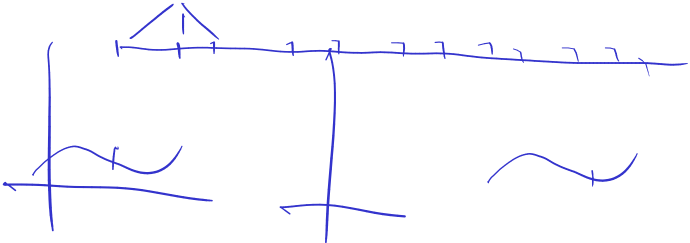
$$f'(\vec{x}) \approx \underbrace{V' V^{-1}}_{D} f(\vec{x})$$

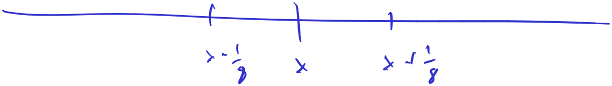
$D =$

Demo: Finite Differences

Demo: Finite Differences vs Noise

Demo: Floating point vs Finite Differences





h

$$4 \cdot f\left(x + \frac{1}{8}\right) - 4 f\left(x + \frac{1}{8}\right) = \frac{f(x+h) - f(x-h)}{2h}$$

Numerical derivatives: why not?

- sensitive to noise
- catastrophic cancellation
- $\|\partial_x\|$

$$\|\partial_x e^{i\alpha x}\|_\infty = \|i\alpha|\, e^{i\alpha x}\|_\infty$$

$$\|e^{i\alpha x}\|_\infty = 1 \qquad | = \alpha$$

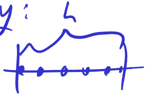
- unbounded operators
- discretizing ∂_x well means suffering

$A \approx \partial_x$

$Ax = b$

error amp: $\kappa(A) = \|A\|\|A^{-1}\|$

Orders of accuracy:

n points $\rightsquigarrow$ 

- poly interp $E(h) = O(h^n)$
- differentiation $E(h) = O(h^{n-1})$
- quadrature $E(h) = O(h^{n+1})$

$(h \to 0)$ order of accuracy

Differencing Order of Accuracy Using Taylor

Find the order of accuracy of the finite difference formula
$f'(x) \approx [f(x+h) - f(x-h)]/2h$.

$$f'(x) - \frac{f(x+h) - f(x-h)}{2h}$$

$$= \cancel{f'(x)} - \frac{1}{2h}\left[\cancel{f(x)} + \cancel{h f'(x)} + \cancel{\frac{h^2}{2} f''(x)} + \frac{h^3}{6} f'''(x) + O(h^4)\right]$$

$$+ \frac{1}{2h}\left[\cancel{f(x)} + \cancel{h f'(x)} + \cancel{\frac{h^2}{2} f''(x)} + \frac{h^3}{6} f'''(x) + O(h^4)\right]$$

$$= \frac{1}{2h}\frac{h^3}{6} f'''(x)$$

Outline

Interpolation Error

If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most n that interpolates f at n distinct points $\{x_i\}$ $(i = 1, \ldots, n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n).$$

Set the error term to be $R(x) := f(x) - p_{n-1}(x)$ and set up an auxiliary function:

$$Y(t) = R(t) - \frac{R(x)}{W(x)}W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n}(t - x_i).$$

Note also the introduction of t as an additional variable, independent of the point x where we hope to prove the identity.

Interpolation Error: Proof cont'd

$$Y(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

- Since x_i are roots of $R(t)$ and $W(t)$, we have $Y(x) = Y(x_i) = 0$, which means Y has at least $n+1$ roots.
- From Rolle's theorem, $Y'(t)$ has at least n roots, then $Y^{(n)}$ has at least one root ξ, where $\xi \in I$.
- Since $p_{n-1}(x)$ is a polynomial of degree at most $n-1$, $R^{(n)}(t) = f^{(n)}(t)$. Thus

$$Y^{(n)}(t) = f^{(n)}(t) - \frac{R(x)}{W(x)} n!.$$

- Plugging $Y^{(n)}(\xi) = 0$ into the above yields the result.

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

- The error bound suggests choosing the interpolation nodes such that the product $|\prod_{i=1}^{n}(x - x_i)|$, is as small as possible. The Chebyshev nodes achieve this.
- Error is zero at the nodes
- If nodes scoot closer together near the interval ends, then

$$(x - x_1)(x - x_2)\cdots(x - x_n)$$

clamps down the (otherwise quickly-growing) error there.

Error Result: Simplified From

Boil the error result down to a simpler form.

Assume $x_1 < \cdots < x_n$.

- $\left|f^{(n)}(x)\right| \leq M$ for $x \in [x_1, x_n]$,
- Set the interval length $h = x_n - x_1$.
  Then $|x - x_i| \leq h$.

Altogether–there is a constant C independent of h so that:

$$\max_x |f(x) - p_{n-1}(x)| \leq CMh^n.$$

For the grid spacing $h \to 0$, we have

$$E(h) = O(h^n).$$

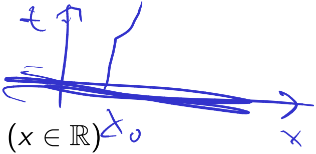
This is called *convergence of order n*.

Demo: Interpolation Error

Outline

Outline

1D Advection Equation and Characteristics



$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Solution?

Solve $u_t + f(u)_x = 0$. $\leftarrow$ $f(u) = au$

Want: $x(t)$ so that $u(x(t),t) = u(x_0, 0)$, where $x(0) = x_0$.

$$\frac{dx(t)}{dt} = f'(u(x(t),t)) \qquad x(0) = x_0$$

$$\frac{du(x(t),t)}{dt} = u_x x'(t) + u_t = u_x f'(u(x(t),t)) + u_t$$

$$= f(u)_x + u_t = 0$$

Hyperbolic conservation law?

characteristics show wave-like behavior also observed on second-order hyperbolic PDEs

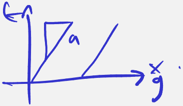
Solving Advection with Characteristics

$$u_t + a u_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Find the characteristic curve for advection.

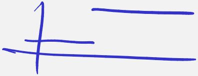
$\frac{dx}{dt} = a \qquad x(0) = x_0 \qquad x(t) = at + x_0$

Generalize this to a solution formula.



$u(t,x) = g(x - at) \leftarrow$

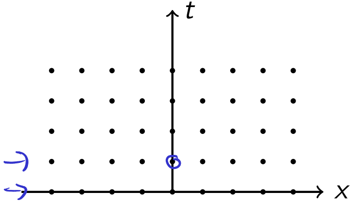
a: advection speed



Does the solution formula admit solutions that aren't obviously allowed by the PDE?

Idea: weaken "derivative"

Finite Difference for Hyperbolic: Idea



$$\{(x_k, t_\ell) : x_k = kh_x, t_\ell = \ell h_t\}$$

If $u(x,t)$ is the exact solution, want

$$u_{k,\ell} \approx u(x_k, t_\ell).$$

Condition at each grid point?

- Pick a stencil for each derivative in the PDE
- Get system of eqns.
- Solve

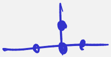
What are explicit/implicit schemes?

Explicit: $\exists$ solution formula

Implicit: you have to solve

Designing Stencils

ETCS:



ITCS:

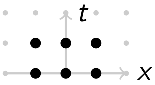
ETFS:

ETBS:

Terminology?

Write out ITCS:

Crank-Nicolson



Crank-Nicolson

Write out Crank-Nicolson: