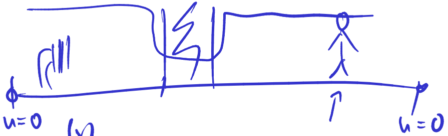
Today

- diss/disp. experiments
- $u_t = -au_x$ ← hyp.
  ↪ $u_t = au_{xx}$ ← parabolic
- nonlinear

Announcements

- HW2 due
- HW3 out later today
- Office Hours AK Fri @ 4:30pm

Heat Equation



Heat equation $(D > 0)$:

$$\begin{aligned} u_t &= Du_{xx}, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) \qquad x \in \mathbb{R}. \end{aligned}$$

Fundamental solution $(g(x) = \delta(x))$:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

$$-(x/2\sqrt{t})^2$$

Why is this a weird model?

Infinite speed of prop

Schemes for the Heat Equation

Cook up some schemes for the heat equation.

Explicit Euler:

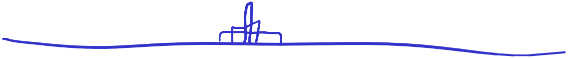
$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} - D\,\frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h_x^2} = 0$$

Implicit Euler:

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} - D\,\frac{u_{k+1,\ell+1} - 2u_{k,\ell+1} + u_{k-1,\ell+1}}{h_x^2} = 0$$

Von Neumann Analysis of Explicit Euler for Heat (1/2)

Let $\lambda = Dh_t/h_x^2$.



$$u_{k,\ell+1} = u_{k,\ell} + \lambda(u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}).$$

$P_h = I$ $\qquad$ $Q_h = \text{tridiag}(\lambda, 1-2\lambda, \lambda)$

$\hat{p}(\varphi) = 1$

$\hat{q}(\varphi) = \lambda e^{-i\varphi} + (1-2\lambda) + \lambda e^{i\varphi} = 1-2\lambda+2\lambda\cos(\varphi)$

$|g| \le 1$

$-1 \le 1+2\lambda(\cos(\varphi)-1) \le 1$

$-2 \le 2\lambda(\cos(\varphi)-1) \le 0$

Von Neumann Analysis of Explicit Euler for Heat (2/2)

$$-2 \le 2\lambda(\cos(\varphi) - 1) \le 0.$$

$-2 \le \cos\varphi - 1 \le 0$

$-2 \le -4\lambda \iff \frac{1}{2} \ge \frac{D h_t}{h_x^2} \iff h_t \le \frac{h_x^2}{2D}$

$h_t = O(h_x^2)$

Comment on the stability region found regarding speeds of propagation.

Explicit scheme propagates at 1 cell/step.

Von Neumann Analysis of Implicit Euler for Heat

Let $\lambda = Dh_t/h_x^2$.

$$u_{k,\ell+1} - \lambda(u_{k+1,\ell+1} - 2u_{k,\ell+1} + u_{k-1,\ell+1}) = u_{k,\ell}$$

$P_h = \text{tridiag}(-\lambda, 1+2\lambda, -\lambda), \quad Q_h = I$

$\hat{p}(\varphi) = 1 + 2\lambda(1-\cos(\varphi)), \quad \hat{q}(\varphi) = 1$

$|s| \le 1 \Leftrightarrow \left|\frac{1}{\hat{p}(\varphi)}\right| \Leftrightarrow 1 \le |1 + 2\lambda(\underbrace{1-\cos\varphi}_{0 \le \cdot \le 2})|$

↪ unconditionally stable

infinite speed of prop.

Does the type of system we need to solve for implicit+parabolic correspond to another PDE?

elliptic

Outline

Outline

Conservation Laws: Recap

$$u_t + f(u)_x = 0,$$ $\int_a^b \; dx$

where u is a function of x and $t \in \mathbb{R}_0^+$.

Rewrite in integral form:

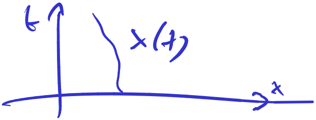
$$\frac{d}{dt}\int_a^b u(x,t)\,dx + f(u(b,t)) - f(u(a,t)) = 0$$

for all a, b

Recall: Characteristic Curve: a function $x(t)$ so that $u(x(t),t) = u(x_0, 0)$.

characteristic speed: $f'(u)$

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f'(u(x(t),t)), \\ x(0) = x_0. \end{cases}$$



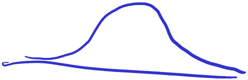
What assumption underlies all this?

differentiability

Burger's Equation

$u_t + f(u)_x = 0 \Rightarrow u_t + f'(u) u_x = 0$

Consider Burgers' Equation:



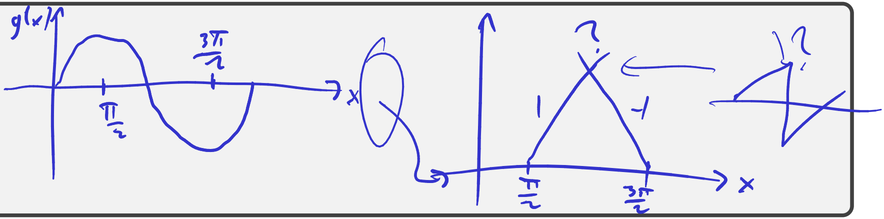
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(x,0) = g(x) = \sin(x). \end{cases}$$

$u_t + u u_x = 0$

Interpret Burger's equation.

$f'(u) = u$

Consider the characteristics at $\pi/2$ and $3\pi/2$.



Weak Solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t))$$

Define a weak solution:

- If u satisfies for almost all (a,b), then u is called a weak solution
- $\varphi \in C_0^1(\mathbb{R}^2)$ ← "compact support"

$$-\int_0^\infty \int_{-\infty}^\infty u\varphi_t + f(u)\varphi_x \, dx\,dt - \int_{-\infty}^\infty g(x)\varphi(x,0)\,dx = 0$$

equivalent

Rankine-Hugoniot Condition (1/2)

Consider: Two C^1 segments separated by a curve $x(t)$ with no regularity.

$$\frac{d}{dt}\left(\underbrace{\int_a^{x(t)} u(x,t)\,dx}_{G_a(x(t),t)} + \int_{x(t)}^{b} u(x,t)\right) + f(u(b,t)) - f(u(a,t)) = 0$$

$$\frac{d}{dt} G_a(x(t),t) = \frac{\partial G_a(x(t),t)}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial G_a}{\partial t}$$

$$= u(x(t),t)\,x'(t) + \int_a^{x(t)} u_t(x,t)\,dx$$